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Some results of precise asymptotics for Lévy processes

Let $\{X(t),\ t \geq 0\}$ be a Lévy processes with $EX(1) = 0$ and $EX^2(1) < \infty$. In this paper, we give two precise asymptotic theorems for $\{X(t),\ t \geq 0\}$.

1. Introduction

Let $Z,\ Z_1,\ Z_2,\ \ldots$ be a sequence of i.i.d. random variables and $S_n = \sum\limits_{i=1}^{n} Z_i,\ n \in \mathcal{N}$. Denote $\log\log t = \ln\ln(t \vee e)$. Gut and Spătaru [1] showed that

Theorem A. Suppose that $EZ = 0,\ EZ^2 = \sigma^2$ and $E[Z^2(\log\log|Z|)^{1+\delta}] < \infty$ for some $\delta > 0$. Let $a_n = O(\sqrt{n}(\log\log n)^{-\gamma})$ for some $\gamma > 1/2$. Then we have

$$\lim_{\varepsilon \downarrow \sigma\sqrt{2}} \sqrt{\varepsilon^2 - 2\sigma^2} \sum_{n \geq 3} \frac{1}{n} P(|S_n| \geq \varepsilon\sqrt{n\log\log n} + a_n) = \sigma\sqrt{2}. \tag{1.1}$$

Theorem B. Suppose that $EZ = 0$ and that $EZ^2 = \sigma^2 < \infty$. Then

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \sum_{n \geq 3} \frac{1}{n\log n} P(|S_n| \geq \varepsilon\sqrt{n\log\log n}) = \sigma^2. \tag{1.2}$$

In this paper, we shall extend the above result to Lévy processes. $\{X(t),\ t \geq 0\}$ with $X(0) = 0$ is called a Lévy process if X has stationary independent increments with sample paths $a.s.$ in $D[0, \infty)$. See Bertoin [5] and Sato [6] for details. It is well-known that if $L(\lambda)$ is the Lévy exponent of X, then

$$Ee^{i\lambda X(t)} = \exp\{-tL(\lambda)\}, \quad \lambda \in \mathcal{R}, \quad t \geq 0.$$

By the famous Lévy-khintchine formula

$$L(\lambda) = -i\gamma\lambda + \frac{A}{2}\lambda^2 + \int_{\mathcal{R}} (1 - e^{i\lambda x} - i\lambda x I(|x| \leq 1))\nu(dx),$$

where γ, A are reals with $A \geq 0$ and ν is a measure on $\mathcal{R}$ satisfying

$$\int_{-\infty}^{\infty} (x^2 \wedge 1)\nu(dx) < \infty.$$

By Theorem 25.3 in Sato [6], $EX^2(1) < \infty$ if and only if

$$\int_{|x|>1} x^2\nu(dx) < \infty,$$

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which is also equivalent to

$$\int_{-\infty}^{\infty} x^2 \nu(dx) < \infty.$$

Throughout, let $\{X(t),\ t \geq 0\}$ be a Lévy process with $EX(1) = 0$ and $EX^2(1) = \sigma^2 < \infty$. The main result of this paper is as follows.

Theorem 1.1. *Let $\{X(t), t \geq 0\}$ be a Lévy process with $EX(1) = 0$ and $EX^2(1) = \sigma^2 < \infty$. Suppose that $E[X^2(1)(\log\log|X(1)|)^{1+\delta}] < \infty$ for some $\delta > 0$. Then*

$$\lim_{\varepsilon \searrow \sigma} \sqrt{\varepsilon^2 - \sigma^2} \int_e^{\infty} \frac{1}{t} P\left(|X(t)| \geq (2\varepsilon - 1)\sqrt{2t\log\log t}\right) dt = \frac{\sqrt{2}}{2}\sigma.$$

Theorem 1.2. *Let $\{X(t), t \geq 0\}$ be a Lévy process with $EX(1) = 0$ and $EX^2(1) = \sigma^2 < \infty$. Then*

$$\lim_{\varepsilon \searrow 0} \varepsilon^2 \int_e^{\infty} \frac{1}{t\log t} P\left(|X(t)| \geq \varepsilon\sqrt{t\log\log t}\right) dt = \sigma^2$$

2. Proof of Theorem 1.1

Let Φ and φ denote the distribution function and the density function, respectively, of the standard normal distribution. Set $\Psi(x) = 1 - \Phi(x) + \Phi(-x)$. There are two main steps for the proof. Firstly, we use $\Psi(x)$ to approach $P(|X(t)|/\sqrt{t} \geq x)$. Secondly, we compute the limit for $\Psi(x)$.

Lemma 2.1. *Let $\{X(t), t \geq 0\}$ be a Lévy process with $EX(1) = 0$ and $EX^2(1) = \sigma^2 < \infty$. With $\Psi(x)$ defined above, we have*

$$\Delta(t) = \sup_x \left| P\left(|X(t)|/\sqrt{t} \geq x\right) - \Psi(x)\right| \to 0, \quad t \to \infty. \tag{2.3}$$

P r o o f. By the central limit theorem,

$$\sup_x \left| P\left(|X(n)|/\sqrt{n} \geq x\right) - \Psi(x)\right| \to 0, \quad n \to \infty. \tag{2.4}$$

Clearly,

$$\sup_{n \leq t < n+1} \left| X(t)/\sqrt{t} - X(n)/\sqrt{n}\right| \leq \sup_{n \leq t < n+1} |X(t) - X(n)|/\sqrt{t} + |X(n)|(1/\sqrt{n} - 1/\sqrt{n+1}). \tag{2.5}$$

Denote $Y_n = \sup_{n \leq t < n+1} |X(t) - X(n)|$, then $\{Y_n,\ n \in \mathcal{N}\}$ is a sequence of i.i.d. random variables. Since $EX^2(1) < \infty$, by Theorem 25.18 in [6], we have $EY_1^2 < \infty$, which is equivalent to $\lim_{n\to\infty} Y_n/\sqrt{n} = 0$, *a.s.*. And $EX(1) = 0$ implies $\lim_{n\to\infty} X(n)/n = 0$ *a.s.* by the Kolmogrov Strong Law of Large Numbers. Then from (2.5), we obtain

$$\sup_{n \leq t < n+1} \left| X(t)/\sqrt{t} - X(n)/\sqrt{n}\right| \to 0,\ a.s. \quad n \to \infty.$$

Thus, there exists $\varepsilon_n \downarrow 0$ such that

$$\sum_{n=1}^{\infty} P\left(\sup_{n \leq t < n+1} \left| X(t)/\sqrt{t} - X(n)/\sqrt{n}\right| > \varepsilon_n\right) < \infty.$$

We have,

$$\begin{aligned} P\left(\left|\frac{X(n)}{\sqrt{n}}\right| > x + \varepsilon_n\right) - P\left(\left|\frac{X(t)}{\sqrt{t}} - \frac{X(n)}{\sqrt{n}}\right| \geq \varepsilon_n\right) &\leq \\ \leq P\left(\left|\frac{X(t)}{\sqrt{t}}\right| > x\right) = P\left(\left|\frac{X(t)}{\sqrt{t}} - \frac{X(n)}{\sqrt{n}} + \frac{X(n)}{\sqrt{n}}\right| > x\right) &\leq \\ \leq P\left(\left|\frac{X(t)}{\sqrt{t}} - \frac{X(n)}{\sqrt{n}}\right| > \varepsilon_n\right) + P\left(\left|\frac{X(n)}{\sqrt{n}}\right| \geq x - \varepsilon_n\right). \end{aligned}$$

Then,

$$\left|P\left(\left|\frac{X(t)}{\sqrt{t}}\right|>x\right)-\Psi(x)\right|\le$$
$$\le\left|P\left(\left|\frac{X(n)}{\sqrt{n}}\right|>x+\varepsilon_n\right)-\Psi(x+\varepsilon_n)\right|+|\Psi(x+\varepsilon_n)-\Psi(x)|+$$
$$+\left|P\left(\left|\frac{X(n)}{\sqrt{n}}\right|>x-\varepsilon_n\right)-\Psi(x-\varepsilon_n)\right|+|\Psi(x-\varepsilon_n)-\Psi(x)|+$$
$$+2P\left(\left|\frac{X(t)}{\sqrt{t}}-\frac{X(n)}{\sqrt{n}}\right|\ge\varepsilon_n\right),$$

which together with (2.4) yields (2.3).

Lemma 2.2. *We have*

$$\lim_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\int_e^\infty\frac{1}{t}\Psi\left((2\varepsilon-1)\sqrt{2\log\log t}\right)dt=\frac{\sqrt{2}}{2}\tag{2.6}$$

P r o o f. By partial integration, we obtain

$$\lim_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\int_e^\infty\frac{1}{t}\Psi\left((2\varepsilon-1)\sqrt{2\log\log t}\right)dt=$$
$$=\lim_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\cdot\frac{1}{(2\varepsilon-1)^2}\int_0^\infty y\Psi(y)e^{\frac{y^2}{2(2\varepsilon-1)^2}}dy=$$
$$=\lim_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\left(-\int_0^\infty\Psi'(y)e^{\frac{y^2}{(2\varepsilon-1)^2}}dy\right)=$$
$$=\lim_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\cdot\frac{2}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{y^2((2\varepsilon-1)^2-1)}{2(2\varepsilon-1)^2}}dy=$$
$$=\lim_{\varepsilon\searrow 1}\sqrt{\frac{\varepsilon+1}{\varepsilon}}\cdot(2\varepsilon-1)(1-\Phi(0))=\frac{\sqrt{2}}{2}.$$

Lemma 2.3. *Put* $b(\varepsilon)=\exp\{\exp\{(\varepsilon^2-1)^{-\frac{3}{2}}\}/\sqrt{\varepsilon^2-1}\}$. *We have, uniformly for all sufficiently small* $\varepsilon-1>0$,

$$\lim_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\int_{b(\varepsilon)}^\infty\frac{1}{t}\Psi\left(\varepsilon\sqrt{2\log\log t}\right)dt=0.\tag{2.7}$$

P r o o f. By putting $y=\varepsilon\sqrt{2\log\log t}$, we get

$$\lim_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\int_{b(\varepsilon)}^\infty\frac{1}{t}\Psi\left(\varepsilon\sqrt{2\log\log t}\right)dt=$$
$$=\lim_{\varepsilon\searrow 1}\frac{2\sqrt{\varepsilon^2-1}}{\sqrt{2\pi}\varepsilon^2}\int_{\varepsilon\sqrt{2[(\varepsilon^2-1)^{-\frac{3}{2}}-\log\sqrt{\varepsilon^2-1}]}}^\infty y\Psi(y)e^{-\frac{y^2}{2}(1-\frac{1}{\varepsilon^2})}dy=$$
$$=\lim_{\varepsilon\searrow 1}2\varepsilon\left(1-\Phi\left(\sqrt{2(\varepsilon^2-1)\log\frac{\exp\{(\varepsilon^2-1)^{-\frac{3}{2}}\}}{\sqrt{\varepsilon^2-1}}}\right)\right)=$$
$$=\lim_{\varepsilon\searrow 1}2\varepsilon\left(1-\Phi\left(\sqrt{2}(\varepsilon^2-1)^{-\frac{1}{4}}\right)\right)=0.$$

We are now prepared to complete the proof of 1.1.

P r o o f of theorem 1.1. Without loss of generality, we assume that $EX^2(1)=1$. At first, by Lemma 2.1, we have

$$\limsup_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\int_e^{b(\varepsilon)}\frac{1}{t}\left|P\left(|X(t)|\ge(2\varepsilon-1)\sqrt{2t\log\log t}\right)-\Psi\left((2\varepsilon-1)\sqrt{2\log\log t}\right)\right|dt\le$$
$$\le\limsup_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\int_e^{b(\varepsilon)}\frac{1}{t}\Delta(t)dt=0.$$

In order to prove the theorem, we only need to prove

$$\lim_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\int_{b(\varepsilon)}^{\infty}\frac{1}{t}P\left(|X(t)|\geq(2\varepsilon-1)\sqrt{2t\log\log t}\right)dt=0.$$

For Lévy process $\{X(t), t\geq 0\}$, we have $\max\limits_{0\leq s\leq 1}|X_s|<\infty$, *a.s.*, which means that for any $0<\varepsilon_0<1$, there exist $a>0$ such that $P(\max\limits_{0\leq s\leq 1}|X_s|\leq a)>\varepsilon_0$ holds. By $\varepsilon>1$ and the stationary independent increments property of Lévy process, for any $n\geq b(\varepsilon)=\exp\left\{\frac{M}{\sqrt{\varepsilon^2-1}}\right\}$ with sufficient large $M>0$ and $n\leq t<n+1$,

$$\begin{aligned}&P\left(|X(n+1)|\geq\varepsilon\sqrt{2n\log\log n}\right)\geq\\&\geq P\left(|X(n+1)|\geq(2\varepsilon-1)\sqrt{2n\log\log n}-a\right)\geq\\&\geq P\left(|X(t)|\geq(2\varepsilon-1)\sqrt{2t\log\log t},\ |X(n+1)-X(t)|\leq a\right)\geq\\&\geq\varepsilon_0 P\left(|X(t)|\geq(2\varepsilon-1)\sqrt{2t\log\log t}\right).\end{aligned}$$

On the other hand, from the proof of Theorem 1 in [1], we know

$$\begin{aligned}&\lim_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\sum_{n>b(\varepsilon)}\frac{1}{n}\left|P\left(|S_n|\geq\varepsilon\sqrt{2n\log\log n}\right)-\Psi\left(\varepsilon\sqrt{2\log\log n}\right)\right|\leq\\&\leq\lim_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\sum_{n>3}\frac{1}{n}\left|P\left(|S_n|\geq\varepsilon\sqrt{2n\log\log n}\right)-\Psi\left(\varepsilon\sqrt{2\log\log n}\right)\right|=0.\end{aligned}$$

Together with (2.7), we have

$$\limsup_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\sum_{n\geq b(\varepsilon)}\frac{1}{n}P\left(|X(n)|\geq\varepsilon\sqrt{2n\log\log n}\right)=0.$$

Then we have,

$$\begin{aligned}&\limsup_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\int_{b(\varepsilon)}^{\infty}\frac{1}{t}P\left(|X(t)|\geq(2\varepsilon-1)\sqrt{2t\log\log t}\right)dt\leq\\&\leq\frac{1}{\varepsilon_0}\limsup_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\sum_{n>b(\varepsilon)}\frac{1}{n}P\left(|X(n+1)|\geq(2\varepsilon-1)\sqrt{2n\log\log n}-a\right)\leq\\&\leq\frac{1}{\varepsilon_0}\limsup_{\varepsilon\searrow 1}\sqrt{\varepsilon^2-1}\sum_{n>b(\varepsilon)}\frac{1}{n}P\left(|X(n+1)|\geq\varepsilon\sqrt{2n\log\log n}\right)=0.\end{aligned}$$

Thus the proof is completed.

3. Proof of Theorem 1.2

We use a similar method as that in section 2.

Lemma 3.4. *We have*

$$\varepsilon^2\int_e^{\infty}\frac{1}{t\log t}\Psi\left(\varepsilon\sqrt{\log\log t}\right)dt=1. \tag{3.8}$$

P r o o f. By putting $y=\varepsilon\sqrt{t\log\log t}$, we get

$$\begin{aligned}\varepsilon^2\int_e^{\infty}\frac{1}{t\log t}\Psi\left(\varepsilon\sqrt{\log\log t}\right)dt&=\varepsilon^2\int_0^{\infty}\frac{2}{\varepsilon^2}y\Psi(y)dy=\\&=\frac{4}{\sqrt{2\pi}}\int_0^{\infty}y\int_y^{\infty}e^{-\frac{x^2}{2}}dxdy=\frac{4}{\sqrt{2\pi}}\int_0^{\infty}e^{-\frac{x^2}{2}}\int_0^x ydydx=\\&=\frac{2}{\sqrt{2\pi}}\int_0^{\infty}x^2e^{-\frac{x^2}{2}}dx=\frac{2}{\sqrt{\pi}}\int_0^{\infty}t^{\frac{1}{2}}e^{-t}dt=\\&=\frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right)=1.\end{aligned}$$

Lemma 3.5. *Put* $c(\varepsilon, m) = \exp\{\exp(m/\varepsilon^2)\}$, *where* $m > 1$. *We have, uniformly for* $\varepsilon \in (0, \infty)$,

$$\lim_{m\to\infty} \varepsilon^2 \int_{c(\varepsilon,m)}^{\infty} \frac{1}{t\log t} \Psi\left(\varepsilon\sqrt{\log\log t}\right) dt = 0.$$

P r o o f. By putting $y = \varepsilon\sqrt{t \log\log t}$, for all ε, we get

$$\varepsilon^2 \int_{c(\varepsilon,m)}^{\infty} \frac{1}{t\log t} \Psi\left(\varepsilon\sqrt{\log\log t}\right) dt = 2\int_{\sqrt{m}}^{\infty} y\Psi(y)dy,$$

and the conclusion follows.

P r o o f of theorem 1.2. Without loss of generality, we assume that $EX^2(1) = 1$. At first, by Lemma 2.1, we have

$$\limsup_{\varepsilon\searrow 0} \varepsilon^2 \int_{e}^{c(\varepsilon,m)} \frac{1}{t\log t} \left| P\left(|X(t)| \geq \varepsilon\sqrt{t\log\log t}\right) - \Psi\left(\varepsilon\sqrt{\log\log t}\right)\right| dt \leq$$
$$\leq \limsup_{\varepsilon\searrow 0} \varepsilon^2 \int_{e}^{c(\varepsilon,m)} \frac{1}{t\log t} \Delta(t)dt = 0.$$

In order to prove the theorem, we only need to prove

$$\lim_{\varepsilon\searrow 0} \varepsilon^2 \int_{c(\varepsilon,m)}^{\infty} \frac{1}{t\log t} P\left(|X(t)| \geq \varepsilon\sqrt{t\log\log t}\right) dt = 0.$$

Similar to the proof in Section 2, we have

$$P\left(|X(n+1)| \geq \varepsilon\sqrt{n\log\log n} - a\right) \geq$$
$$\geq P\left(|X(t)| \geq \varepsilon\sqrt{t\log\log t},\ |X(n+1) - X(t)| \leq a\right) \geq$$
$$\geq \varepsilon_0 P\left(|X(t)| \geq \varepsilon\sqrt{t\log\log t}\right).$$

Then by Proposition 3.4 in [1], we have

$$\limsup_{\varepsilon\searrow 0} \varepsilon^2 \int_{c(\varepsilon,m)}^{\infty} \frac{1}{t} P\left(|X(t)| \geq \varepsilon\sqrt{t\log\log t}\right) dt \leq$$
$$\leq \frac{1}{\varepsilon_0} \limsup_{\varepsilon\searrow 0} \varepsilon^2 \sum_{n>c(\varepsilon,m)} \frac{1}{n} P\left(|X(n+1)| \geq \varepsilon\sqrt{n\log\log n} - a\right) \leq$$
$$\leq \frac{1}{\varepsilon_0} \limsup_{\varepsilon\searrow 0} \varepsilon^2 \sum_{n>c(\varepsilon,m)} \frac{1}{n} P\left(|X(n+1)| \geq \frac{\varepsilon}{2}\sqrt{n\log\log n}\right) = 0.$$

Thus the proof is completed.

4. Relative

Heyde [3] showed that

$$\lim_{\varepsilon\searrow 0} \varepsilon^2 \sum_{n=1}^{\infty} P\left(|S_n| \geq \varepsilon n\right) = EZ^2,$$

if and only if $EZ = 0$ and $EZ^2 < \infty$. Since then, many other results have been found; see, for instance, Chen [7], Gut and Spătaru [1], Gut and Spătaru [2] and Hu and Su [8]. We list some results in [2] as follows: if $EX^2 = \sigma^2 < \infty$ and $EX = 0$, then

$$\lim_{\varepsilon\searrow 0} -\frac{1}{\log\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n} P\left(|S_n| \geq \varepsilon n^{1/p}\right) = \frac{2p}{2-p}, \tag{4.9}$$

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2(r-p)/(2-p)} \sum_{n=1}^{\infty} n^{r/p-2} P\left(|S_n| \geq \varepsilon n^{1/p}\right) = \frac{p}{r-p} E|N|^{2(r-p)/(2-p)}, \tag{4.10}$$

$$\lim_{\varepsilon \searrow 0} \varepsilon^{2\delta+2} \sum_{n=1}^{\infty} \frac{(\log n)^{\delta}}{n} P\left(|S_n| \geq \varepsilon \sqrt{n \log n}\right) = \frac{E|N|^{2\delta+2}}{\delta+1} \sigma^{2\delta+2}, \tag{4.11}$$

where $1 \leq p < 2$, $1 < p < r < 2$ and $0 \leq \delta \leq 1$ respectively, and N denotes a standard normal random variable.

Remark 1. Similarly, (4.9), (4.10) and (4.11) can also be extended to Lévy processes.

Remark 2. The authors are grateful for the referee's careful review and suggestions, which improved our results.

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ABSTRACT

Let $\{X(t),\ t \geq 0\}$ be a Lévy processes with $EX(1) = 0$ and $EX^2(1) < \infty$. In this paper, we give two precise asymptotic theorems for $\{X(t),\ t \geq 0\}$.

Key words: *precise asymptotic, Lévy process, stable process, Fuk-Nagaev type inequality.*