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Billiards in unbounded domains reversing the direction of motion of a particle

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We consider the Euclidean space $\mathbb{R}^2$ with orthogonal coordinates x, y and billiards in $\mathbb{R}^2 \setminus \Omega$, where the set Ω satisfies the conditions:

(a) Ω lies in the upper half-plane $\{y \geqslant 0\}$ and contains the half-plane $\{y > 1\}$,

(b) $\partial\Omega$ is a non-self-intersecting piecewise-smooth curve of class C^2.

Let $S^1_+ = \{\vec{v} = (v_1, v_2) \in \mathbb{R}^2 : |\vec{v}| = 1,\ v_2 \geqslant 0\}$ be the upper unit semicircle and for any $x \in \mathbb{R}$, $\vec{v} \in S^1_+$ consider a billiard particle with coordinates $(x(t), y(t)) = (x, 0) + \vec{v}\,t$ for $t \leqslant 0$. If the particle is reflected a finite number of times at regular points of $\partial\Omega$ and then moves freely, we denote its final velocity by $\vec{v}^{\,+}_\Omega(x, \vec{v}\,)$. The function $\vec{v}^{\,+}_\Omega$ is thus defined on a subset of $\mathbb{R} \times S^1_+$ and takes values in $S^1_- := \{\vec{v} = (v_1, v_2) : |\vec{v}| = 1,\ v_2 \leqslant 0\}$. It turns out that $\vec{v}^{\,+}_\Omega$ is defined on a dense open subset of $\mathbb{R} \times S^1_+$ and is continuous. The proof is not difficult and essentially repeats the arguments used in the proof of Lemma 1.7.1 in [1].

The main object of this note is the construction of a family of sets Ω_ε, $\varepsilon > 0$, satisfying conditions (a) and (b) and such that

$$\vec{v}^{\,+}_{\Omega_\varepsilon}(x, \vec{v}) \text{ converges in measure to } -\vec{v} \text{ as } \varepsilon \to 0^+ \tag{1}$$

on any Borel subset of $\mathbb{R}\times S^1_+$ of finite Lebesgue measure. To this end, consider the set $E_\varepsilon = \{(x, y) : y \geqslant 0,\ y \geqslant \varepsilon(|x| - \varepsilon),\ x^2/(1 + \varepsilon^2) + y^2 \leqslant 1\}$, a figure bounded above by an arc of an ellipse and below by the major axis and sloping lines passing through the foci, and the truncated cone $K_\varepsilon = \{(x, y) : y \geqslant 0,\ y \geqslant \varepsilon(x - \varepsilon),\ y \geqslant -\varepsilon(x + \varepsilon)\}$. Let $E^{k,b}_\varepsilon$, $K^{k,b}_\varepsilon$, $k > 0$, be the sets obtained from E_ε, K_ε by the homothety with centre $(0, 0)$ and coefficient k and a subsequent translation by the vector $(b, 0)$. We now give an inductive definition of sets $\mathscr{E}^{(n)}_\varepsilon$ and $\mathscr{K}^{(n)}_\varepsilon$, $n \in \mathbb{N}$. Put $\mathscr{E}^{(1)}_\varepsilon = \bigcup_{m\in\mathbb{Z}} E^{1,2m\sqrt{1+\varepsilon^2}}_\varepsilon$, $\mathscr{K}^{(1)}_\varepsilon = \bigcup_{m\in\mathbb{Z}} K^{1,2m\sqrt{1+\varepsilon^2}}_\varepsilon$. Assume that the sets $\mathscr{E}^{(n)}_\varepsilon$ and $\mathscr{K}^{(n)}_\varepsilon$ are already defined for some n. Consider pairs (k, b) such that the distance between the sets $\mathscr{K}^{(n)}_\varepsilon$ and $E^{k,b}_\varepsilon$ is equal to k. (By the distance between two sets we mean the greatest lower bound of the distances between an element of one set and an element of the other.) The set of all such pairs is countable, they have the same first component $k = k_n$, and the set of second components $b = b_{nm}$, $m \in \mathbb{Z}$, forms an infinite two-sided arithmetic progression. Put $\mathscr{E}^{(n+1)}_\varepsilon = \mathscr{E}^{(n)}_\varepsilon \cup \bigl(\bigcup_{m\in\mathbb{Z}} E^{k_n,b_{nm}}_\varepsilon\bigr)$, $\mathscr{K}^{(n+1)}_\varepsilon = \mathscr{K}^{(n)}_\varepsilon \cup \bigl(\bigcup_{m\in\mathbb{Z}} K^{k_n,b_{nm}}_\varepsilon\bigr)$. The set $\mathscr{E}^{(n)}_\varepsilon$ is invariant under translation by the vector $(2\sqrt{1+\varepsilon^2}, 0)$ and each of the sets $I_n := ([0,\, 2\sqrt{1+\varepsilon^2}] \times \{0\}) \setminus \mathscr{E}^{(n)}_\varepsilon$ is the union of a finite number of intervals of common length $|I_n|$ forming a decreasing geometric progression as n increases. Choose n_ε in such a way that $|I_{n_\varepsilon}| \leqslant \varepsilon$ and put $\Omega_\varepsilon = \{(x, y) : y > 0\} \setminus \mathscr{E}^{(n_\varepsilon)}_\varepsilon$. It is clear that Ω_ε satisfies conditions (a) and (b).

Let N_ε be the set of values $(x, \vec{v}) \in \mathbb{R} \times S^1_+$ such that either $v_2 \leqslant \varepsilon/\sqrt{1+\varepsilon^2}$ or $x + 2m\sqrt{1+\varepsilon^2} \in \bar{I}_{n_\varepsilon}$ for some $m \in \mathbb{Z}$. If $(x, \vec{v}) \notin N_\varepsilon$, then the billiard particle in $\mathbb{R}^2 \setminus \Omega_\varepsilon$ with coordinates $(x, 0) + \vec{v}\,t$ for $t \leqslant 0$ is reflected exactly once at a point of the upper boundary of one of the figures $E^{k_n,b_{nm}}_\varepsilon$ and its subsequent velocity is equal to $\vec{v}^{\,+}_{\Omega_\varepsilon}(x, \vec{v}) = -\vec{v} + O(\varepsilon)$, and this bound is uniform with respect to x and $\vec{v}$. On the other hand, for any Borel set $M \subset \mathbb{R}\times S^1_+$ of finite Lebesgue measure, the Lebesgue measure of the intersection $M \cap N_\varepsilon$ is $O(\varepsilon)$. This proves (1).

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This construction can be generalized to the case of higher dimensions. For example, in the Euclidean space $\mathbb{R}^3$ with orthogonal coordinates x, y, z we fix an $\varepsilon > 0$ and consider the set $\mathrm{O}_\varepsilon = \{(x,y,z) : 0 \leqslant z \leqslant \sqrt{1-x^2-y^2},\ z \geqslant \varepsilon \max\{|x|-\varepsilon, |y|-\varepsilon\}\}$ bounded above by the upper unit hemisphere and below by its equatorial plane and four inclined planes. Let $\mathrm{O}_\varepsilon^{k,x,y}$, $k>0$, be the set obtained from O_ε by the homothety with centre the origin and coefficient k and subsequent translation by the vector $(x,y,0)$. As in the two-dimensional case, the set $\mathscr{O}_\varepsilon$ is constructed as the union of a countable family of pairwise disjoint sets of the form $\mathrm{O}_\varepsilon^{k,x,y}$ (with different triples k, x, y in general): it is invariant under translation by the vectors $(2,0,0)$ and $(0,2,0)$ and is such that the area of the planar set $([0,2]\times[0,2]\times\{0\}) \setminus \mathscr{O}_\varepsilon$ is at most ε. The set $\Omega_\varepsilon^{(3)} = \{(x,y,z) : z > 0\} \setminus \mathscr{O}_\varepsilon$ is contained in the half-space $\{z \geqslant 0\}$, contains the half-space $\{z > 1\}$ and has piecewise smooth boundary. For almost all $(x,y,\vec{v}) \in \mathbb{R}^2 \times S_+^2$, where $S_+^2 = \{\vec{v} = (v_1,v_2,v_3) \in \mathbb{R}^3 : |\vec{v}| = 1,\ v_3 \geqslant 0\}$, a billiard particle in $\mathbb{R}^3 \setminus \Omega_\varepsilon^{(3)}$ with coordinates $(x,y,0) + \vec{v}\,t$ for $t \leqslant 0$ undergoes a finite number of reflections and then moves freely with velocity $\vec{v}^{\,+}_{\Omega_\varepsilon^{(3)}}(x,y,\vec{v})$. The function $\vec{v}^{\,+}_{\Omega_\varepsilon^{(3)}}$ so defined converges in measure to $-\vec{v}$ in any subset of $\mathbb{R}^2 \times S_+^2$ of finite Lebesgue measure.

Let us clarify the meaning of our results by giving an example. A vertical stream of particles of density ρ and fixed cross-section $[a,b]$ falls on an unbounded body in $\mathbb{R}^2$, so that the first configuration coordinate of the falling particles lies between a and b. The initial position of the body coincides with a set Ω satisfying conditions (a) and (b). The body can turn around the first coordinate, and the probability of turning through an angle $\varphi \in [-\pi/2, \pi/2]$ is defined as the probability measure ν on $S_+^1 = \{(\cos\varphi, \sin\varphi),\ \varphi \in [-\pi/2, \pi/2]\}$. The mathematical expectation of the vertical component of the force of pressure of the stream on the body is equal to $R(\Omega) = \rho \int_{S_+^1} \int_a^b [1 - (\vec{v}_\Omega^{\,+}(x/v_2, \vec{v}), \vec{v})]\, dx\, d\nu(\vec{v})$. It is required to find $\sup_\Omega R(\Omega)$.

The family $\{\Omega_\varepsilon\}$ of sets constructed above yields a solution of this problem. Firstly, we have the relation $\sup_\Omega R(\Omega) = 2\rho(b-a) = \lim_{\varepsilon\to 0+} R(\Omega_\varepsilon)$. Secondly, the mathematical expectation of the force of pressure on the vertical half-plane $\Omega^0 := \{(x,y) : y > 0\}$ is equal to $R(\Omega^0) = 2\rho(b-a) \int_{S_+^1} v_2^2\, d\nu(\vec{v})$. In particular, if ν is a measure uniformly distributed on S_+^1, then the quantity $R(\Omega^0) = \rho(b-a)$ is equal to half the maximum value of R.

In the 3-dimensional case, we similarly arrive at the relation $\sup_\Omega R(\Omega) = 2\rho\sigma = \lim_{\varepsilon\to 0+} R(\Omega_\varepsilon^{(3)})$, where σ is the cross-sectional area of the stream and ρ is its density. On the other hand, for the upper half-space $\Omega_0^{(3)} := \{(x,y,z) : z > 0\}$, we have $R(\Omega_0^{(3)}) = 2\rho\sigma \int_{S_+^2} v_2^2\, d\mu(\vec{v})$, where μ is a probability measure on S_+^2. In the case when μ is uniformly distributed on S_+^2, the quantity $R(\Omega_0^{(3)}) = \frac{2}{3}\rho\sigma$ is equal to one third of the maximal value of R.

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Bibliography

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