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LOCALIZATION FOR CONTINUOUS RANDOM SCHRÖDINGER OPERATORS IN THE SEMICLASSICAL LIMIT

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§0. Introduction

In this paper, we want to present (without proof) some recent results concerning localization for certain semiclassical random Schrödinger operators. The operators that we consider are perturbations of periodic Schrödinger operators. These perturbations are essentially of two types: the first model is of an alloy-type that is, in each of its wells, a periodic potential is perturbed by some compactly supported function whose size is given by a random variable. In the second model, we do not change the shape of the wells, but randomly distort the lattice. Instead of putting a potential well at each point of a lattice, we will move the well in a small set around the lattice point, the position of the well in this set being chosen randomly. Finally we consider both types of perturbations at the same time in a model that will be called the mixed model. All the random variables to be used in our models will be chosen independent identically distributed (i.i.d).

Such hamiltonians are used in solid state physics (see, [Li-Mat, L-Gr-P]). They describe the behaviour of an electron in an alloy (model 1), in a pure crystal with a disturbed lattice structure (model 2) or in an alloy with a disturbed lattice structure. (mixed model).

In the last 20 years, a quite extensive mathematical litterature about random operators in general and random Schrödinger operators in particular has been developed (see, e.g., the monographs [P-Fi, C-L]).

Our models enjoy translation symmetry, and the random variables they involve are i.i.d. Thus, they are endowed with nice ergodic properties; in particular, this ensures that there exists Σ, a closed subset of $\mathbb{R}$, such that, with probability 1 (i.e for almost every realization of our random variables), Σ is the spectrum of our operator. In the same way, there exists Σ_{pp}, Σ_{ac} and Σ_{sc}, closed subsets of $\mathbb{R}$, such that, with probability 1, Σ_{pp} is the pure point spectrum, Σ_{ac}, the absolutely continuous spectrum, and Σ_{sc} is the singular continuous spectrum of our operator (see, e.g., [P-Fi, C-L] and the references therein).

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The point of interest for us here is localization. Let I be some subset of $\mathbb{R}$. We say that an operator under study shows localization in I if $\Sigma_{\text{ac}} \cap I = \varnothing$, $\Sigma_{\text{sc}} \cap I = \varnothing$ and $\Sigma \cap I \neq \varnothing$. We say that it shows exponential localization in I if it shows localization and if, moreover, each eigenfunction associated to an eigenvalue from I is exponentially decreasing at infinity.

The localization phenomena has first been studied by physicists (cf [An, A-An-T]) for some discrete Schrödinger operators. Mathematically, localization has only been shown much more recently, mainly for 1-dimensional models (continuous or discrete) (see, e.g., [G-Mo-P, Ko-Si, C], and references therein), and for discrete models on a lattice (see, e.g., [F-Sp, M-S, vD-K] and [Ai-Mo], and references therein).

In 1-d, localization has been shown to occur at all energies for many hamiltonians whereas, in the higher dimensional discrete models, localization is proved to happen either for large disorder (i.e. large random perturbation) or for large energies (i.e near the fluctuation edges of the almost sure spectrum).

In the d-dimensional continuous case, except for the study of one model (see [H-M] and [Ko-Si]), there were no results until very recently. In the present paper we use semiclassical analysis and the techniques developed for discrete random systems to show rigorous results on localization for continuous systems in dimension larger than 1.

Finally, we point out that, more recently, localization was proved (in a non semi-classical context) for continuous d-dimensional random hamiltonians (cf [Co-His, Kl4]).

§1. The first model

Consider the following Schrödinger operator acting on $L^2(\mathbb{R}^d)$:

$$P = -h^2\Delta + V, \tag{1.1}$$

where V is a potential such that

$$V \text{ is } \mathbb{Z}^d\text{-periodic, } V \in \mathcal{C}^\infty(\mathbb{R}^d),\ V \geq 0,\ V^{-1}(\{0\}) = \mathbb{Z}^d, \text{ and Hess } V(0) \text{ is positive definite.} \tag{H.1.1}$$

P is a lower semibounded, selfadjoint operator with domain $H^2(\mathbb{R}^d)$. From the Floquet theory it follows that $\sigma(P)$, the spectrum of P, is made of bands of absolutely continuous spectrum.

Let $[i_h, s_h]$ be the first band of $\sigma(P)$. Under assumption (H.1.1), we know that, for h small enough, there exists $a(h) > 0$ such that

$$\begin{cases} \limsup_{h\to 0}(h\log|s_h - i_h|) \leq -S_0 \\ |a(h)| + h|\log(a(h))| \to 0 \quad \text{when } h \to 0 \\ \sigma(P) \cap ([i_h, s_h] + [-2a(h), 2a(h)]) = [i_h, s_h] \end{cases} \tag{1.2}$$

(see [Kl1] and references therein; S_0 is defined in the following way: if d denotes the Agmon distance determined by V, that is the distance given by the metric $V(x)dx$ (cf [He-Sj]), then $S_0 = \inf_{\alpha \neq \beta} d(\alpha, \beta) > 0$).

Let δV be a compactly supported function such that

$$\delta V \geq 0,\ \delta V(0) = \|\delta V\|_\infty = 1,\ \delta V \in C_0^\infty(\mathbb{R}^d), \text{ and } \delta V \text{ is nicely supported in a neighborhood of } 0 \tag{H.1.2}$$

(for a more precise statement, see [Kl2])

Let g be the density of a probability distribution supported on $[-1, 1]$ such that there exists $\epsilon_0 > 0$ and $\rho_0 > 0$ satisfying

$$\int_{-\infty}^{+\infty} \sup_{|v| \leq \epsilon} |g(u+v) - g(u)| du \leq \left(\frac{\epsilon}{\epsilon_0}\right)^{\rho_0} \tag{1.3}$$

for $0 \leq \epsilon \leq \epsilon_0$. Let $(t_\gamma)_{\gamma \in L}$ be i.i.d random variables with common density g_h defined, for $u \in \mathbb{R}$, by the formula

$$g_h(u) = \frac{1}{\widetilde{a}(h)} g\left(\frac{u}{\widetilde{a}(h)}\right), \text{ where } \widetilde{a}(h) \text{ satisfies } 0 < \widetilde{a}(h) \leq a(h) \text{ and } \lim_{h \to 0} h \log \widetilde{a}(h) = 0. \tag{1.4}$$

We define the following random operator:

$$P_t = P + \sum_{\gamma \in L} t_\gamma \delta V_\gamma,$$

where $\delta V_\gamma(x) = \delta V(x - \gamma)$. For any realization of t, P_t is a lower semibounded, selfadjoint operator with domain $H^2(\mathbb{R}^d)$.

Theorem 1. *Let P_t be defined as above; we assume* (H.1.1)–(H.1.2). *There exists $h_0 > 0$ such that, for $h \in (0, h_0)$, with probability* 1,

(a) $\sigma(P_t) \cap [i_h - 2a(h), s_h + 2a(h)] \neq \varnothing$;

(b) *the spectrum of P_t in $[i_h - 2a(h), s_h + 2a(h)]$ is pure point;*

(c) *if φ is an eigenvector of P_t associated with an eigenvalue in $[i_h - 2a(h), s_h + 2a(h)]$, then there exists $C(\varphi, h) > 0$ such that*

$$|\varphi(x)| \leq C(\varphi, h) e^{-\frac{h_0}{h}|x|}$$

for any $x \in \mathbb{R}^d$.

Sketch of the proof. The proof of this result may be found in [Kl2]. First, using a semiclassical reduction theorem "à la Helffer-Sjöstrand" (cf [He-Sj, Ca] or [Kl1]), we show that the operator P_t restricted to the energy interval $[i_h - 2a(h), s_h + 2a(h)]$ is unitarily equivalent to some infinite random matrix acting on $l^2(\mathbb{Z}^d)$. The coefficients of this matrix can be controlled with a good precision. Moreover, the main random terms are on the diagonal (as the random perturbations are located in the wells of the periodic potential). Then the method of Fröhlich-Spencer (cf [F-Sp, vD-K]) can be extended so as to show localization for this kind of operators.

On the heuristic level, localization at this energy level may be understood in the following way: the first band is of size $s_h - i_h$, and the random perturbation of size $\widetilde{a}(h)$. $s_h - i_h$ is exponentially small with h, and $\widetilde{a}(h)$ is much larger (cf (1.4)); so, the disorder is much stronger than the unperturbed operator. We are in a large disorder case.

§2. The second model

We pick a function $\theta \in C_0^\infty((-\frac{1}{2}, \frac{1}{2})^d)$ such that

$$-1 \leq \theta \leq 0,\ \theta^{-1}(-1) = \{0\}, \text{ and } -1 \text{ is a nondegenerate minimum of } \theta. \tag{H.2.1}$$

On $L^2(\mathbb{R}^d)$, we consider

$$Q = -h^2\Delta + \theta. \tag{1.5}$$

The standard semi-classical analysis (see, for example [He-Sj] or [He] and references therein) implies that, if $\mu(h)$ denotes the infimum of the spectrum $\sigma(Q)$ of Q, then there exists $h_0 > 0$, $C_0 > 0$ such that, for $h \in (0, h_0)$,

(i) $\mu(h)$ is a simple eigenvalue and $\mu(h) \to -1$ when $h \to 0$;
(ii) $\sigma(Q) \cap [\mu(h) - 2C_0 h, \mu(h) + 2C_0 h] = \{\mu(h)\}$.

Picking $\omega = (\omega_\alpha)_{\alpha\in\mathbb{Z}^d} \in \Omega^{\mathbb{Z}^d}$, we introduce the function $V_\omega(x) = \sum_{\alpha\in\mathbb{Z}^d} \theta(x - \alpha - \omega_\alpha)$.

Consider the following Schrödinger operator acting on $L^2(\mathbb{R}^d)$:

$$P_\omega = -h^2\Delta + V_\omega. \tag{1.6}$$

Since V_ω is bounded for $\omega \in \Omega^{\mathbb{Z}^d}$, P_ω is a selfadjoint operator semibounded from below and having domain $H^2(\mathbb{R}^d)$.

Let g be density of a distribution in $\mathbb{R}^d$ (i.e., $g \geq 0$ and $\int_{\mathbb{R}^d} g = 1$), and let G be the closed support of g. Without loss of generality, we may assume that $0 \in G$. Suppose that G and θ satisfy the following conditions:

1) G is such that the Agmon distance determined by $(1+V_\omega)$ is uniformly nondegenerate for $\omega = (\omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ with $\omega_\gamma \in G$.
2) A well (of V_ω) can serve as a nearest neighbor (in the Agmon distance sense) of at most one other well.
3) The Agmon distance between two wells depends smoothly on the position of the wells (at least if the wells are close to be nearest neighbors). (H.2.2)
4) g satisfies the same regularity condition as in §1 (rewritten for functions on $\mathbb{R}^d$).

Remark. a) Precise formulations of these assumptions may be found in [Kl3]. Some of them may be relaxed.

b) One can show that the above assumptions are fulfilled if θ is spherically symmetric, and G and supp(θ) are sufficiently small. or $0 < h < h_0$, let $I(h_0,h) = \mu(h) + [-C_0 h, -e^{-\frac{S_0+h_0}{h}}] \cup [e^{-\frac{S_0+h_0}{h}}, C_0 h]$, where S_0 is the infimum of the Agmon distance (determined by $(1+V_\omega)$) between two distinct wells (ω runs over all possible realizations). Our assumptions imply that $S_0 > 0$.

Theorem 2. *Let P_ω be defined as above; we assume that* (H.2.1)–(H.2.2) *are satisfied. Then there exists $h_0 > h_0' > 0$ such that, for $h \in (0, h_0')$, with probability* 1, *we have*

(a) $\sigma(P_\omega) \cap I(h_0,h) \neq \varnothing$;
(b) *the spectrum of P_ω in $I(h_0,h)$ is pure point*;
(c) *if φ is an eigenfunction of P_ω associated with an eigenvalue in $I(h_0,h)$, then there exists $C(h,\varphi) > 0$ such that, for $x \in \mathbb{R}^d$,*

$$|\varphi(x)| \leq C(h,\varphi) e^{-\frac{h_0'}{h}|x|}.$$

Remark. Using the ergodicity of the family of operators P_ω, we can show that there exists some constant $c > 0$ such that, with probability 1, for h small enough,

$$\|P_{\omega|F_\omega} - \mu(h)\| \geq c \cdot h^{1-\frac{d}{2}} e^{-\frac{S_0}{h}}.$$

So, Theorem 2 says that, as h goes to 0, most of the band we look at becomes localized.

Sketch of the proof. The proof of this theorem as well as that of Theorem 3 may be found in [Kl3]. Again, as in Theorem 1, using the nondegeneracy of the Agmon distance given by $(1+V_\omega)$, one can prove a semiclassical reduction theorem. But in this case, since all the wells are of the same size and shape, the randomness comes from the interaction coefficients, i.e. from the fluctuation of the Agmon distance between the wells. Semiclassical analysis (as developed in [He-Sj]) allows us to control these distances; then the extension of the Fröhlich-Spencer method constructed in [Kl2] can be applied to show localization for this model.

§3. The mixed model

Let θ be chosen as in §2 (i.e. θ satisfies (H.2.1)), and let g_Ω be a distribution density on $\mathbb{R}^d$ with closed support G_Ω such that G_Ω and θ satisfy (H.2.2). Let g_T be a distribution density on $\mathbb{R}$ supported on $[-1,1]$ and satisfying (1.3). Pick a function $\widetilde{a}(h)$ such that $0 < \widetilde{a}(h) < \frac{1}{2}C_0 h$ for $h \in (0, h_0)$. Suppose that the limit of $h \log a(h)$ exists as h tends to 0. We define $g_{h,T}$ as g_T scaled by (1.4).

Let $(t_\gamma, \omega_\gamma)_{\gamma\in\mathbb{Z}^d}$ be i.i.d random variables with common distribution density $g_{h,T} \otimes g_\Omega$. Consider the following random Schrödinger operator:

$$P_{t,\omega} = -h^2\Delta + \sum_{\gamma\in\mathbb{Z}^d}(1+t_\gamma)\theta(x-\gamma-\omega_\gamma) = -h^2\Delta + V_{t,\omega}. \tag{1.7}$$

Then, for every realization of (t,ω), $P_{t,\omega}$ is a semibounded selfadjoint operator with domain H^2.

We now define two different asymptotic regimes in ω and t depending on $\widetilde{a}(h)$. If $\lim_{h\to 0} h\log\widetilde{a}(h) < -S_0$, we say that we are in regime (1); then, for $0 < h < h_0$, we put $I_1(h_0,h) = \mu(h) + [-C_0 h, -e^{-\frac{S_0+h_0}{h}}] \cup [e^{-\frac{S_0+h_0}{h}}, C_0 h]$. We are in regime (2) if $\lim_{h\to 0} h\log\widetilde{a}(h) \geq -S_0$; in this case, for $0 < h < h_0$, we put $I_2(h_0,h) = \mu(h) + [-C_0 h, C_0 h]$ (here S_0 is defined as in §2).

Theorem 3. *Let θ, g_Ω and $g_{a,T}$ be chosen as above. Let $(t_\alpha,\omega_\alpha)_{\alpha\in\mathbb{Z}^d}$ be independent identically distributed random variables having $g_{a,T}\otimes g_\Omega$ as common distribution density. Let $P_{t,\omega}$ be the operator defined by* (1.7).

Then, in regime (k) for $k=1$ or 2, there exists $h_0 > h_0' > 0$ such that, for $h \in (0, h_0')$, with probability 1, *we have*

(a) $\sigma(P_{t,\omega}) \cap I_k(h_0,h) \neq \varnothing$;
(b) *the spectrum of $P_{t,\omega}$ in $I_k(h_0,h)$ is pure point;*
(c) *if φ is an eigenfunction associated with an eigenvalue E of $P_{t,\omega}$ lying in $I_k(h_0,h)$, then there exists $C(h,\varphi) > 0$ such that, for $x \in \mathbb{R}^d$,*

$$|\varphi(x)| \leq C(h,\varphi)e^{-\frac{h_0'}{h}|x|}.$$

Remark. In the above formulation we deal with only two different asymptotic regimes. In fact, there are three regimes, namely:

(1) when $\lim_{h\to 0} h\log\widetilde{a}(h) < -S_0$,
(2) when $\lim_{h\to 0} h\log\widetilde{a}(h) = -S_0$,
(3) when $\lim_{h\to 0} h\log\widetilde{a}(h) > -S_0$.

In regime (1), the effect produced by the random variables $(\omega_\alpha)_{\alpha\in\mathbb{Z}^d}$ dominates the one created by $(t_\alpha)_{\alpha\in\mathbb{Z}^d}$. So, only the $(\omega_\alpha)_{\alpha\in\mathbb{Z}^d}$ are responsible for localization. Hence we can only prove localization for a part of the band.

In regime (3), it is the effect produced by the random variables $(t_\alpha)_{\alpha\in\mathbb{Z}^d}$ that dominates; the whole band gets localized.

In regime (2), both of the effects are of the same order of magnitude, but none of them alone is strong enough to localize the whole band. It is the summing up of both effects that localizes the band.

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