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Evgenii (Eugene) Borisovich Dynkin (obituary)

The prominent mathematician Evgenii (Eugene) Borisovich Dynkin, who taught in the Faculty of Mechanics and Mathematics at Moscow State University from 1948 to 1968 and was a professor at Cornell University from 1977, passed away on 14 November 2014, at the age of 90. He was internationally famous due to his work in two different fields of mathematics, the theory of Lie groups and probability theory.

He was born on 11 May 1924 in Leningrad (now St. Petersburg). His father was a lawyer and his mother was a dentist, both born in Belorussia. In 1935, in the wake of the Kirov assassination, authorities sent the family into exile in Kazakhstan. In 1937 his father was arrested and later perished in the Gulag. In 1940 Evgenii Dynkin graduated with a brilliant record from the secondary school in Aktyubinsk (now Aktobe), even skipping one year, and enrolled in the Faculty of Mechanics and Mathematics at Moscow State University (MSU).

When the Second World War came to the Soviet Union, Dynkin moved to Tula Oblast, where his mother had been working since the end of her exile, and then they were evacuated to Perm. He was not conscripted into the army because of his strong nearsightedness and consequences of the Potts' disease he had suffered in childhood. He continued his education at Perm University, where he stayed until the end of 1943. S. A. Yanovskaya, a professor in the Faculty of Mechanics and Mathematics at MSU who was in Perm at that time, played a significant beneficial role in his life. She recommended him to I. M. Gelfand, and when Dynkin returned to Moscow, Gelfand invited him to participate in his seminar.

As a fourth-year student, Dynkin was asked by Gelfand to prepare a talk on the structure and classification of semisimple Lie algebras based on the works of E. Cartan, Weyl, and van der Waerden. In the process Dynkin devised 'systems of simple roots' which enabled him to significantly simplify the classification and which subsequently changed drastically the entire picture of the theory of semisimple Lie algebras. This was his first research work (with regard to the time of its completion).

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As a fifth-year student Dynkin participated in the seminar of A. N. Kolmogorov, and together with his fellow participant Kolya (Nikolai) Dmitriev he solved a problem posed by Kolmogorov on the characteristic numbers of a stochastic matrix. This made up his second paper submitted for publication.

On graduating from the university in 1945, he enrolled for graduate studies, with Kolmogorov as his advisor. He defended his Ph.D. thesis ahead of time, and in 1948 was hired by the Department of Probability Theory of the Faculty of Mechanics and Mathematics at MSU as a senior lecturer. A year later he became a docent, and after defending his D.Sc. thesis in 1951 was soon promoted to professor. Both of his theses were concerned with Lie groups and algebras. In 1951 these works, which subsequently came to be regarded as classical, earned him the Prize of the Moscow Mathematical Society.

From 1955 onwards, Dynkin made a complete switch to probability theory. His works in this field were mostly concerned with the theory of Markov processes, where he is regarded as one of the founders. His fundamental monograph *Markov processes* [17], published in 1963, remained a standard source of information in the area for many years to come and is still regarded as a classical exposition of the theory. Special mention should be made of Dynkin's last cycle of papers, which was devoted to superprocesses, Markov processes related to non-linear partial differential equations. His works on game theory and mathematical economics, written in the 1960s and 70s are less well known to the mathematical community. Only in recent years has their importance become clear, especially due to recently discovered applications of 'Dynkin games' to financial mathematics.

Dynkin was a brilliant lecturer and a remarkable teacher. While a professor in the Department of Probability Theory, he also gave courses in algebra and mathematical analysis. His lectures and seminars were never boring: he knew how to get the listeners' attention by his extraordinary presentation of the material and interesting problems, and he always had plenty of students. The seminar on Lie groups that he founded is still active under the supervision of his students È. B. Vinberg and A. L. Onishchik, who now have research schools of their own. At different periods of time such mathematicians as V. I. Arnold, F. A. Berezin, F. I. Karpelevich, I. I. Piatetski-Shapiro, and others who subsequently developed into well-known mathematicians participated in his seminar. The list of Dynkin's students in probability theory includes many of the leaders of the next generation: N. V. Krylov, M. B. Malyutov, S. A. Molchanov, A. V. Skorokhod, M. I. Freidlin, R. Z. Khasminskii, and others. Quite a number of mathematicians, some of whom later became famous, mention Dynkin in connection with their first research results, even though he was not necessarily their formal advisor.

While still a student at Perm University, Dynkin organized a student study group for three other students from MSU who were also in Perm. On returning to Moscow he became a supervisor of a section in a mathematics study group for secondary school students. The work of this section in 1945–1947 formed a basis for the book *Mathematical conversations*,[1] written together with V. A. Uspenskii, a former participant of that study group. It is now a classical text in the genre of popular educational reading.

[1] *Editor's note*. Translated as: E. B. Dynkin and V. A. Uspenskii, *Mathematical conversations*: *Multicolor problems*, *problems in the theory of numbers*, *and random walks*, Heath, Boston 1963.

In the early 1960s, in connection with the school education reform of that time, schools with an emphasis on mathematics were organized in the USSR. Dynkin played a prominent role in their early history. In 1963, with the aid of his graduate students and other young mathematicians, he organized the After-Hours Mathematical School at Moscow school no. 2, and the next year, with the support of the school principal V. F. Ovchinnikov, he organized there (in 1964–1966) classes for mathematically gifted students in the 9th and 10th form. This is where Dynkin's organizational talents were clearly exhibited. Using a cleverly designed system of lectures, practical exercises, and problem-solving competitions, they created a unique competitive atmosphere.

A certain independence of Dynkin's views did not go over well with the authorities, and in the spring of 1968 he was forced to leave MSU. After being unemployed for some time, he eventually became a researcher in the Department of Mathematics at the Central Institute for Economics and Mathematics of the USSR Academy of Sciences. Even at those times he maintained a home seminar for a selected group of students.

In 1976 Dynkin took the decision to emigrate. He accepted an offer from Cornell University (Ithaca, NY) and obtained a prestigious position. He worked there for 33 years and retired only at the age of 86. Up to his last days he maintained close connections with his Russian students and colleagues.

For several decades Dynkin interviewed well-known mathematicians from Russia and other countries. A main concern of his last years was to systematize this extensive collection of audio- and video-recordings, putting them down in written form, and translating Russian interviews into English. His collection is now available at a Cornell University site.[2]

In 1985 Dynkin was elected to the USA National Academy of Sciences, and in 1995 the Moscow Mathematical Society, of which Dynkin had been a vice-president in 1964–1971, elected him an honourary member. He also was a Doctor Honoris Causa of several universities, including the Independent University of Moscow.

Dynkin authored more than 200 research papers, eight research monographs, and three popular educational books. He continued his active research until very late in life: his last paper was published in 2013. In 2000 the American Mathematical Society published a volume of his selected works, with commentaries by experts [51].

In this paper we survey Dynkin's works on the theory of Lie groups and probability theory. His works on mathematical economics were reviewed by I. V. Evstigneev in [51]. We include in the bibliography only Dynkin's works cited in the text: as a rule, these are not his first research announcements, but rather later papers which contain full proofs.

1. Dynkin's papers on Lie groups

1.1. Simple root systems. The classification of complex semisimple Lie algebras obtained by W. Killing and E. Cartan at the end of the 19th century is known to be based on the weight decomposition of a Lie algebra with respect to a Cartan subalgebra. Thus, for each semisimple Lie algebra $\mathfrak{g}$ we construct a *root system* Δ, which is a finite system of non-zero vectors in a Euclidean space E (of dimension equal to the rank of $\mathfrak{g}$), with the following properties:

[2] http://dynkincollection.library.cornell.edu.

(K0) Δ spans E;
(K1) if $\alpha \in \Delta$, then $k\alpha \in \Delta$ if and only if $k = \pm 1$;
(K2) if $\alpha, \beta \in \Delta$, then $(\alpha|\beta) =: 2(\alpha,\beta)/(\alpha,\alpha) \in \mathbb{Z}$;
(K3) for each $\alpha \in \Delta$ the system Δ is invariant under the reflection

$$R_\alpha \colon x \mapsto x - (\alpha|x)\alpha.$$

The group W generated by the reflections R_α is called the *Weyl group* of the system Δ (or of the algebra $\mathfrak{g}$). The condition (K3) means that W preserves the lattice generated by the roots.

The algebra $\mathfrak{g}$ is uniquely determined by a root system of it, and in a Euclidean space each system of vectors satisfying (K0)–(K3) is a root system of some semisimple Lie algebra.

By the *direct sum* of root systems we mean their union, regarded as a system of vectors in the direct sum of the corresponding Euclidean spaces. A root system is said to be *irreducible* if it cannot be represented as a direct sum of two root systems. Irreducible root systems correspond to simple Lie algebras. The root system of a simple Lie algebra is uniquely determined up to a dilation.

All root systems were known before Dynkin, but no uniform way of constructing them was available. Dynkin's idea [2] was to replace the root system by a simpler object which uniquely determines it. Namely, he proposed to divide the roots into positive roots and negative (opposite to positive) roots in the sense of the lexicographic ordering defined by a basis in E, and to look at the positive roots that cannot be represented as a sum of two positive roots. He called these the *simple* roots and proved that the system Π of simple roots forms a basis of E and has the following property:

(Π) $(\alpha|\beta) \in \mathbb{Z}_-$ for any distinct roots $\alpha, \beta \in \Pi$.

Up to an action of the Weyl group the system of simple roots is independent of the choice of the basis used to define the lexicographic ordering, and the root system Δ can be uniquely recovered from it. Moreover, each basis Π of E with the property (Π) is a system of simple roots of some root system Δ, and Δ is irreducible if and only if Π is. Thus, the classification of root systems (and therefore of semisimple complex Lie algebras) reduces to a fairly easy problem in linear algebra.

It follows from (Π) that the angle between two simple roots has the form $\pi - \pi/m$, where $m \in \{2, 3, 4, 6\}$, and the ratio of the squares of their lengths is equal to 1, 2, or 3 if $m = 3, 4$, or 6, respectively. Dynkin proposed a description of a system of simple roots in terms of the graph with vertices corresponding to simple roots such that two vertices corresponding to roots α and β are joined by a simple, double, or triple edge if $m = 3, 4$, or 6, respectively. In addition, if roots in a connected component of the graph have different lengths, then the vertices corresponding to the shorter roots were coloured black. (Instead of colouring vertices, authors later started orienting the multiple edges). These graphs, later called *Dynkin diagrams*, are in essence a modification of Coxeter diagrams, used by Coxeter in his classification of finite linear groups generated by reflections, without connecting this with root systems. A root system is irreducible if and only is its Dynkin diagram is connected.

It is not an overstatement to say that systems of simple roots (and the Dynkin diagrams used in their description) revolutionized the theory of semisimple Lie groups and algebras. Not just the classification of semisimple complex Lie algebras themselves, but also the description of their linear representations, automorphisms, subalgebras, real forms, and other related objects are natural to formulate using the language of simple roots. For example, an irreducible linear representation of a semisimple Lie algebra is determined by its highest weight Λ, which in turn is determined by its 'numerical marks' $(\alpha|\Lambda)$ on the Dynkin diagram, and these marks can be arbitrary non-negative integers. The outer automorphism group of a semisimple Lie algebra is naturally isomorphic to the automorphism group of its Dynkin diagram [5]. Classes of conjugate parabolic subalgebras (subalgebras containing a maximal solvable subalgebra) are in a natural bijective correspondence with subsets of the system of simple roots [76].

1.2. Semisimple subalgebras of semisimple Lie algebras. In 1950–1952, in connection with the Lie problem of classifying primitive continuous transformation groups, Dynkin carried out the enormous programme of classifying maximal connected Lie subgroups of semisimple complex Lie groups, or equivalently, of classifying maximal subalgebras of semisimple complex Lie algebras [7], [8]. These results made up his D.Sc. thesis.

V. V. Morozov, in his D.Sc. thesis of 1943 [87], had already proved that if a maximal subalgebra of a semisimple Lie algebra is not semisimple, then it is a parabolic subalgebra. In this way he reduced the classification of maximal subalgebras of semisimple Lie algebras to that of semisimple maximal subalgebras. On the other hand, the investigation of semisimple subalgebras of arbitrary semisimple Lie algebras can easily be reduced to that of semisimple subalgebras of simple Lie algebras.

For the classical simple complex Lie groups $G = \mathrm{SL}(V), \mathrm{SO}(V), \mathrm{Sp}(V)$ (where V is a complex vector space) Dynkin's answer [7] was as follows.

Let $H \subset G$ be a semisimple maximal connected Lie subgroup of the group G. If H is a reducible linear group, then G is $\mathrm{SO}(V)$ or $\mathrm{Sp}(V)$, while H is the connected component of the stabilizer of some non-degenerate subspace U of V. If H is irreducible but not simple, then it is a tensor product of two classical linear groups: $H = \mathrm{SL}(U) \otimes \mathrm{SL}(W)$, $\mathrm{SO}(U) \otimes \mathrm{SO}(W)$, $\mathrm{Sp}(U) \otimes \mathrm{Sp}(W)$, or $\mathrm{SO}(U) \otimes \mathrm{Sp}(W)$, where $U \otimes W = V$.

In the most interesting case of a simple irreducible group H, Dynkin established the following surprising result. He showed that, as a rule, an irreducible simple connected linear group $H \subset \mathrm{SL}(V)$ is a maximal connected Lie subgroup either of $\mathrm{SL}(V)$, or — if it preserves a non-degenerate bilinear form — of $\mathrm{SO}(V)$ or $\mathrm{Sp}(V)$. The complete list of exceptional cases consists of 4 series and 14 particular linear groups. In all these cases he explicitly indicated in [7] a connected simple linear group which contains H as a proper subgroup and is distinct from the corresponding classical linear group.

Also in [7] he proved some general theorems on linear representations of semisimple Lie algebras which are of independent interest. In particular, he found there some irreducible components of a product of irreducible representations with highest weights Λ and M for a semisimple Lie algebra that are different from the highest (Cartan) component, whose highest weight is $\Lambda + M$. More precisely, let $H(\Lambda, M)$ be the set of highest weights of all the irreducible components in this product. We can introduce a partial ordering in this set by writing $\lambda \geqslant \mu$ if $\lambda - \mu$ is a linear combination of simple roots with non-negative integer coefficients. Then $\Lambda + M$

is the only maximal element of $H(\Lambda, M)$. In [7] Dynkin found all the maximal elements of the set $H(\Lambda, M) \setminus \{\Lambda + M\}$. It turned out that these elements are just weights of the form

$$\Lambda + M - \alpha_1 - \cdots - \alpha_k,$$

where $\alpha_1, \ldots, \alpha_k$ are distinct simple roots such that no two consecutive elements of the sequence $\Lambda, \alpha_1, \ldots, \alpha_k, M$ are orthogonal, while elements that are not consecutive are orthogonal, with the possible exception only of the pair (Λ, M). The corresponding irreducible components occur with multiplicity 1 in the decomposition (as does the component with highest weight $\Lambda + M$), and their highest vectors can be written out.

In his next paper [8], Dynkin listed all the semisimple subalgebras of the exceptional simple Lie algebras and distinguished the maximal subalgebras among them. Apart from this and many other useful results, [8] contained a large body of factual information which had been obtained by calculations (by hand); it became a reference source for many mathematicians. Here we describe the main ideas and results in that paper.

A subalgebra of a semisimple Lie algebra $\mathfrak{g}$ is said to be *regular* if it can be normalized by means of a Cartan subalgebra. In particular, a subalgebra containing a Cartan subalgebra (for instance, any parabolic subalgebra) is regular. With each subset $\Pi_1 \subset \Pi$ of the system of simple roots of an algebra $\mathfrak{g}$ we can associate a semisimple regular subalgebra such that Π_1 is a system of simple roots for it. Such subalgebras are sometimes called *Levi subalgebras* since they are the maximal semisimple subalgebras of the corresponding parabolic subalgebras.

In [8] Dynkin proposed a simple and elegant method for finding all semisimple regular subalgebras of maximum rank in a simple Lie algebra $\mathfrak{g}$.[3] It is based on considering a so-called *extended system* $\widetilde{\Pi}$ *of simple roots*, obtained by adding a lowest root to the system Π of simple roots of $\mathfrak{g}$. The system $\widetilde{\Pi}$ also satisfies the condition (Π), but is no longer linearly independent: a certain linear relation with positive integer coefficients holds for it (such relations have an important role in the theory of semisimple Lie algebras, for instance, in the description of their finite-order inner automorphisms). As in the case of the description of a system of simple roots by its Dynkin diagram, we can describe an extended system of simple roots using an *extended Dynkin diagram*.

By removing an arbitrary vertex from the extended Dynkin diagram, we obtain the Dynkin diagram of some semisimple (but not simple in general) regular subalgebra of $\mathfrak{g}$ of maximum rank whose system of simple roots is obtained by removing a root from $\widetilde{\Pi}$. This subalgebra may coincide with $\mathfrak{g}$ (for instance, if we have removed the additional root), but such cases should be left out of consideration. Extending one of the connected components of the resulting Dynkin diagram in this way and again removing a vertex of the extended diagram, we now obtain the Dynkin diagram of a smaller semisimple regular subalgebra of maximum rank. Dynkin's result says that by continuing this process, we eventually obtain all the semisimple regular subalgebras of maximum rank. Clearly, we can only obtain maximal subalgebras in the first step, and it can be shown that these are actually obtained if and only if we have removed a root with prime coefficient in the linear relation mentioned above.[4]

[3] This method was implicitly contained in the Borel–de Siebenthal paper [61].

Each semisimple regular subalgebra of a semisimple Lie algebra $\mathfrak{g}$ is a semisimple regular subalgebra of maximum rank in some semisimple Levi subalgebra. In this way we obtain a list of all the semisimple regular subalgebras of the given algebra. However, we must also find out which of these are conjugate by means of an inner automorphism. This was resolved in [8] using direct calculations. Five years prior to his death Dynkin returned to Lie groups and wrote the paper [57] in conjunction with his scientific 'descendant' A. N. Minchenko. There they proposed a beautiful general method for the solution of this problem, based on 'enhanced' Dynkin diagrams which they introduced. Even more important, they used these diagrams in a general method for determining inclusions (up to conjugation) between semisimple regular subalgebras.

After treating the case of regular subalgebras, Dynkin passed over to arbitrary subalgebras. He called a subalgebra of a semisimple Lie algebra $\mathfrak{g}$ an *R-subalgebra* if it lies in a proper regular subalgebra, and otherwise he called it an *S-subalgebra*. In a certain sense, S-subalgebras are an analogue of irreducible linear Lie algebras. From the characterization of reductive linear subalgebras as subalgebras of fixed points of diagonalizable subgroups of the adjoint group (the group of inner automorphisms) of $\mathfrak{g}$ [61] it follows that all the S-subalgebras are semisimple subalgebras that have a trivial centralizer in the group of inner automorphisms of $\mathfrak{g}$.

It is clear that each semisimple subalgebra is an S-subalgebra of some semisimple regular subalgebra. Thus, the description of all semisimple subalgebras of semisimple Lie algebras reduces (modulo the problem of determining conjugate subalgebras) to the description of all semisimple S-subalgebras of simple Lie algebras.

Simple three-dimensional subalgebras make up a significant portion of semisimple subalgebras of exceptional simple Lie algebras (Dynkin called them 'three-term subalgebras', and the modern terminology for them is '$\mathfrak{sl}_2$-triples'). Dynkin showed that such a subalgebra is determined up to conjugation by a semisimple element of it (normalized in a certain special way), which he called the *characteristic* of the subalgebra. We can assume that the characteristic belongs to a fixed Cartan subalgebra and is a dominant element in it, in the sense that all simple roots take non-negative values on it. Dynkin established the a priori fundamental property of a characteristic: simple roots can take values 0, 1, or 2 on it (under the above convention). After that he first found all three-dimensional simple S-subalgebras of simple Lie algebras (the corresponding list is fairly short), and then he used the classification of semisimple regular subalgebras to find all three-dimensional simple subalgebras of exceptional simple Lie algebras. In addition, he calculated the dimensions of irreducible components of the restriction of the adjoint representation of the ambient Lie algebra to these three-dimensional subalgebras.

In view of the Morozov–Jackobson theorem [86], [75] that any nilpotent element of a semisimple Lie algebra is included in a simple three-dimensional subalgebra,[5] and Konstant's theorem [78] that this three-dimensional subalgebra is uniquely determined up to conjugation by means of the centralizer of the nilpotent element under consideration,[6] Dynkin's results can be interpreted as a classification of the nilpotent elements of exceptional simple Lie algebras, though Dynkin himself did

[4]It was erroneously claimed in [8] that all the subalgebras obtained at the first step are maximal.

[5]See [89] for a history of the proof of this theorem.

[6]This result was also proved by Morozov in his unpublished thesis [87]; Morozov's proof can be found in [89].

not pose this question. Subsequently, Dynkin's method was successfully used for classification of the nilpotent elements of more general linear representations of simple Lie algebras [98], [97].

The most complicated and laborious part of [8] was the classification of semisimple S-subalgebras of rank > 1 in exceptional simple Lie algebras. There Dynkin used all the results proved earlier in his paper. Simultaneously, he found all the inclusion relations between semisimple S-subalgebras. Then he used the above scheme to obtain a classification of all semisimple and, finally, all semisimple maximal subalgebras of the exceptional simple Lie algebras.

Still, it should be mentioned that while Dynkin classified the semisimple regular, maximal, and S-subalgebras up to conjugacy, he classified all the remaining subalgebras only up to linear conjugacy. (Two subalgebras of a Lie algebra $\mathfrak{g}$ are *linearly conjugate* if their images under any linear representation of $\mathfrak{g}$ are conjugate in the corresponding full linear Lie algebra). This gap was fixed by Minchenko in the recent paper [85], where he showed that, as a rule, a linear conjugacy class is also a conjugacy class, and he listed all the exceptions.

The results in [8] served as a model for similar results on subgroups of simple algebraic groups over algebraically closed fields of positive characteristic or of simple finite groups of Lie type, although the last group of results required different methods for their proof.

One of the important tools used by Dynkin in [8] was the concept of the index of a homomorphism $\varphi\colon \mathfrak{h} \to \mathfrak{g}$ between simple Lie algebras. Under the assumption that the invariant scalar products in $\mathfrak{g}$ and $\mathfrak{h}$ are normalized so that the scalar square of the highest root (or the corresponding dual root, if we use more invariant terms) is 2, the *index* j_φ of the homomorphism φ is defined by

$$(\varphi(x), \varphi(y)) = j_\varphi(x, y) \quad \text{for any } x, y \in \mathfrak{h}.$$

The index is a positive integer and is multiplicative with respect to composites of homeomorphisms. For the index of an irreducible linear representation $\varphi\colon \mathfrak{g} \to \mathfrak{sl}_n(\mathbb{C})$ with highest weight Λ, Dynkin proved the formula

$$j_\varphi = \frac{n}{\dim \mathfrak{g}}(\Lambda, \Lambda + g),$$

where g is the sum all positive roots of $\mathfrak{g}$.

He used indices in [8], first, to distinguish between isomorphic but not conjugate subalgebras and, second, to show that some embeddings of simple Lie algebras are impossible. The concept of index later proved to be useful in the investigation of generic stabilizers in linear representations of semisimple Lie groups [58], [67].

1.3. The topology of compact Lie groups. A cycle of Dynkin's papers was devoted to the topology of compact Lie groups. At that time it was known from a theorem of Hopf (but also connected with Samelson and others) that the rational cohomology algebra $H^*(G, \mathbb{Q})$ of a connected compact Lie group G is a Grassmann algebra whose number of generators is equal to the rank of G, and that any homogeneous basis of the graded space $P^*(G, \mathbb{Q})$ of primitive cohomology classes can be taken as a system of free generators of this algebra. For the classical compact Lie groups, free generators of this algebra were found by L. S. Pontryagin. Dynkin [10] established an analogous theorem for the weak integral cohomology ring $H^*(G)$

(the quotient of the integral cohomology ring by the torsion ideal), thereby refining Hopf's result. He proposed a general method for constructing a basis of the graded group $P^*(G)$ of primitive weak integral cohomology classes and showed that for the classical groups his method leads to the cohomology classes found by Pontryagin.

Each homomorphism $\Phi\colon H \to G$ of connected compact Lie groups induces a homomorphism $\Phi^*\colon P^*(G) \to P^*(H)$ of graded groups. It is important to know the latter in order to calculate the cohomology of the homogeneous space $G/\Phi(H)$. Using the connection between primitive cohomology classes and polynomial invariants of the adjoint representation, Dynkin [11] proposed a method for calculating the homomorphism Φ^* (which was later refined by I. Z. Rozenknop [91]). Moreover, he proved that an irreducible representation Φ of a connected semisimple compact Lie group H into a classical compact linear group G is determined by Φ^* up to composition with an automorphism of G.

Note that if G and H are simple Lie groups, then $P^3(G) = H^3(G)$ and $P^3(H) = H^3(H)$ are cyclic groups, and the homomorphism $\Phi^3\colon P^3(G) \to P^3(H)$ is determined by an integer which coincides (up to sign) with the index of the homomorphism $d\Phi\colon \mathfrak{h} \to \mathfrak{g}$ between the corresponding Lie algebras (see the definition of the index above).

1.4. Other works. In one of his first papers [3] Dynkin discovered a remarkably simple method for calculating the coefficients in the Campbell–Hausdorff formula. Recall that this formula expresses the formal series of $\log(e^x e^y)$ for non-commuting variables x and y in terms of multiple commutators of these variables, and it is one of the ways to prove the fundamental fact in Lie group theory: that a local Lie group is uniquely determined by its tangent Lie algebra. The coefficients in this formula can be calculated recursively, but no explicit formulae for them were known at the time. Dynkin discovered that they could be easily found using the following general algebraic fact, which he established.

For a given degree-m homogeneous polynomial in the non-commuting variables $x_1, \dots, x_n$, assume that it can be represented as a linear combination of multiple commutators of these variables. Then such a representation can be obtained simply by replacing each product $x_{i_1} x_{i_2} \cdots x_{i_m}$ by $m^{-1}[[\dots[x_{i_1}, x_{i_2}], \dots], x_{i_m}]$!

This enabled Dynkin to generalize Lie's theory to local Banach groups and Lie algebras over an arbitrary complete normed field of characteristic zero [4]. These results made up his Ph.D. thesis.

In the 1960s he also wrote several papers on Brownian motion in symmetric spaces, in which he combined his two lines of research, Lie groups and probability theory. We will mention these papers in the next section.

2. Dynkin's works on probability theory

2.1. Early work in probability theory. Dynkin took an interest in probability already in his student years. His first paper [1], a joint work with Dmitriev, solved a problem posed by Kolmogorov in his seminar on the characteristic numbers of a stochastic matrix. At that time both the authors were fifth-year students in the Faculty of Mechanics and Mathematics at MSU. After defending his Ph.D. thesis, Dynkin was hired by the Department of Probability Theory (then headed by Kolmogorov) in the same faculty. He gave lecture courses in probability theory, algebra, and analysis, and he participated in the work of seminars on probability theory and mathematical statistics. Among his papers of this period we mention [6], which was concerned with sufficient statistics. It it Dynkin found criteria for the existence of necessary and sufficient statistics in terms of the dimension of a certain auxiliary linear space; if this dimension is larger than the sample size, then there exist no non-trivial sufficient statistics. Almost 30 years later Dynkin returned to this subject in [36], where he linked these results with later results on the boundary behaviour of Markov processes.

We also mention [12], where he investigated the behaviour of sums of independent random variables when the terms are non-negative and have infinite expectation.

2.2. Markov processes: foundations. Several authors have written papers on Dynkin's role in the first steps of the theory of Markov processes, for instance, P.-A. Meyer [84] and A. A. Yushkevich, who published two excellent papers, [100] and especially [101]. Understandably, here we shall have to repeat much of what these authors have already written.

Before the mid-1950s the modern concept of a Markov process was non-existent. Some classes of Markov processes had been investigated mostly using analytic methods, either in connection with second-order differential equations (Kolmogorov) or using operator semigroups (Feller). The first investigations of the properties of sample paths of a process in some sense were initiated in the late 1940s for processes with a countable state space, and were first received with little enthusiasm. Yushkevich wrote in [100] that as long ago as 1951 Kolmogorov gave another, purely analytic proof of some of J. L. Doob's results and mentioned that his proofs were simpler than Doob's, where measures in function spaces were used.

Yushkevich points out that Dynkin's interest in Markov processes arose when he was giving a lecture course on the subject. In [9], his first paper in this direction, he presented a criterion for the existence of a Markov process with continuous sample paths in terms of its transition function $p(s,x;t,\Gamma)$. Namely, if for any $\varepsilon>0$ the limit

$$\lim_{\delta\to 0}\ \sup_{t:s<t<s+\delta} p(s,x;t,\overline{U_\varepsilon(x)})$$

is equal to zero and the convergence is uniform in x, then there exists a version of this process which with probability 1 is right-continuous and without discontinuities of the second kind (here $\overline{U_\varepsilon(x)}$ is the complement of the ε-neighbourhood of x). Furthermore, if the expression under the limit sign decays like $o(\delta)$, then the process can be made continuous. A similar result was independently proved by J. Kinney [77] the next year (Kinney's continuity criterion was the same as in [9], but the criterion involving the absence of discontinuities of the second kind was slightly weaker). Still, the corresponding results are known in the theory as the Dynkin–Kinney criterion.

By 1954 Dynkin was concentrating almost completely on the theory of Markov processes. He organized a seminar on the theory which attracted undergraduate and graduate students, and gradually he became one of the founders of the modern theory of Markov processes and one of the leaders of the Moscow school in probability. He published two monographs, [14] and [17], on the theory of Markov processes, which presented his own results and those of other participants in the seminar, including A. D. Wentzell, V. A. Volkonskii, I. V. Girsanov, L. V. Seregin, M. I. Freidlin, R. Z. Khasminskii, M. G. Shur, and Yushkevich. Here we focus on

Dynkin's key ideas, concepts, and constructions, which greatly influenced the general development of the theory (see [14] and [17]).

2.2.1. *The definition of a Markov process.* The birth of the theory of Markov processes was associated with the names of A. A. Markov, L. Bachelier, N. Wiener, Kolmogorov, and A. Ya. Khintchine. For a long time the theory was developing around the transition function and related operators acting in function spaces. A systematic treatment of the properties of sample paths of the process began with Doob [65]. At that time a Markov process was understood to be a stochastic process with the Markov property: conditional independence of the past and the future for the known present (a so-called Markov process in the sense of Doob). However, for a well-developed theory it was necessary to consider simultaneously processes corresponding to different initial states and initial times. These processes had to be compatible, in particular to facilitate working with the conditional distributions for the known present. Also, processes with a random killing time arose naturally in a number of examples. Taking all this into account, Dynkin defined an (inhomogeneous) Markov process as a family of probability measures $P_{s,x}$ on the same space $(\Omega, \mathscr{F})$ of elementary events. The definition involves the killing time ζ and σ-algebras $\mathscr{F}_t^s$, which are interpreted as collections of events which are observable on the time interval $[s,t]$, and the Markov property holds with respect to these σ-algebras.[7] Homogeneous (in the time) Markov processes are an important special case, and their definition also involves the shift operator θ_t.

2.2.2. *The characteristic operator.* In the description of the local behaviour of a Markov process authors usually employed the infinitesimal operator of the operator semigroup connected with the transition function, that is, the limit

$$Af(x) = \lim_{t \to 0} t^{-1}(P_x f(\xi_t) - f(x)).$$

However, this operator characterizes the local behaviour of the process in time rather than in space. In its place Dynkin proposed the characteristic operator

$$\mathfrak{A}f(x) = \lim_{\varepsilon \to 0} \frac{P_x f(\xi_{\tau_\varepsilon}) - f(x)}{P_x \tau_\varepsilon},$$

defined in terms of the first exist time τ_ε from the ε-neighbourhood of the point x, so that in fact he replaced the limit procedure with respect to time by a limit procedure with respect to space (here and in what follows, P_x is the expectation calculated under the condition that the process starts at x). The characteristic operator turned out to be a much more natural object: for instance, it enabled one to give a probabilistic equivalent to the (analytic) theory of one-dimensional diffusion which Feller had developed a short time previously.

2.2.3. *Strong Markov property.* In its simplest form the strong Markov property means that for a certain class of random times τ the stochastic process $\tilde{\xi}_t = \xi_{\tau+t}$ is also a Markov process and has the same transition function as the original process. Here the class of admissible times τ, the so-called Markov times (or stopping times), includes the first arrival times $\tau_\Gamma = \inf\{t \colon \xi_t \in \Gamma\}$, where Γ is a subset of the state

[7] The σ-algebras $\mathscr{F}_t^s$ contain the σ-algebras generated by the sample paths of the process on the intervals $[s,t]$, but do not necessarily coincide with them.

space. For processes with discrete time the strong Markov property is easy to deduce from the ordinary Markov property. For continuous time the situation is much more complicated.

The strong Markov property had often been used in the theory without any substantiation whatsoever. One classical example here is the so-called reflection principle in the analysis of the properties of the maximum of a Brownian motion. The first rigorous result in this direction can be found in Doob's paper [65], where the strong Markov property was established for a class of Markov chains with continuous time. In [66] he extended the strong Markov property to n-dimensional Brownian motions. In addition, he mentioned in [65] and [66] that this was a special case of a much more general result, which, however, he never stated or published.

In its contemporary form the strong Markov property was proved by Dynkin and Yushkevich [13] for right-continuous Feller processes[8] in an arbitrary metric space. In particular, they introduced in [13] such objects as the σ-algebra $\mathscr{F}_\tau$ of events, which describes the 'past' of the process before the time τ. At the same time, similar results for more narrow classes of processes or in a more restrictive form were obtained by G. A. Hunt [72], D. Ray [90], K. L. Chung [63], and R. Blumenthal [60].

Seventeen years later Dynkin returned to this topic in [33].

2.2.4. *Standard processes.* It is clear that the Markov processes form a very broad class which includes plenty of ugly examples. It would be natural to ask about properties of a Markov process enabling one to eliminate unwelcome examples without losing anything important. As an answer, Dynkin suggested the concept of a standard process. This is a (homogeneous) right-continuous strong Markov process without discontinuities of the second kind which takes values in a locally compact Hausdorff space with the second axiom of countability. Moreover, it is assumed that the σ-algebras $\mathscr{F}_t = \mathscr{F}_t^0$ are right-continuous with respect to t, in the sense that $\mathscr{F}_s = \bigcap_{t>s} \mathscr{F}_t$. Finally, if (τ_n) with $\tau_n < \tau$ and $\tau_n \uparrow \tau$ is a monotonically increasing sequence of Markov times, then $\xi_{\tau_n} \to \xi_\tau$ with probability 1 (so-called left quasi-continuity). A similar place in the Western literature was occupied by Hunt processes, which are also left-continuous at the killing time of the process (the value of a standard process is not defined at the killing time, and the corresponding property is meaningless, whereas a Hunt process is also defined at the killing time). Much later, in the mid-1970s, Ray processes were defined for which left quasi-continuity was replaced by the so-called moderate Markov property.

2.2.5. *Natural topology and excessive functions.* The notion of fine topology goes back to H. Cartan [62], who defined it as the weakest topology in which all subharmonic functions are continuous. Later Doob [66] showed that if a point x in the plane does not lie in the closure of a set A in the Cartan topology, then the time τ_A of the arrival at A of a Brownian motion starting at x is strictly positive with probability 1. For standard Markov processes a set A is said to be finely open if, with probability 1, a Markov process starting from any point in A does not reach the complement of A in an infinitesimal time. In [17] a base for the fine topology is constructed and it is proved that a function is finely continuous if and only if with probability 1 it is right-continuous on sample paths of the process (the last result is due to Girsanov and Shur).

[8] A Markov process ξ_t is called a Feller process if the operators $P_t f(x) = P_x f(\xi_t)$ take continuous functions to continuous functions.

2.2.6. *Additive functionals.* An additive functional is a family of random variables A_{st}, $s < t$, which are measurable with respect to the σ-algebras $\mathscr{F}_t^s$ and satisfy $A_{st} + A_{tu} = A_{su}$ for $s < t < u$. A simple example here is the functional of integral type

$$A_{st} = \int_s^t f(u, \xi_u)\, du,$$

where f is some function. Functionals of integral type had been considered before, but the general definition is due to Dynkin [15]. We can interpret a non-negative additive functional as a time scale depending on the sample path of the process. In particular, this covers a local time on a subset of the state space and many other examples. Also in these days, additive functionals remain an important tool in the theory of Markov processes. For example, they are used to the full in studying the properties of superprocesses (see [45], [47], [48]).

2.2.7. *Transformations of Markov processes.* Many authors have considered transformations that, for a given Markov process, enable one to construct another Markov process with required properties. The list of such transformations includes random time change, change of measure, killing, the h-transformation, and many others. Dynkin proposed (α, ξ)-subprocesses, which combine killing and change of measure. Roughly speaking, the multiplicative functional α is responsible for killing, and ξ for change of measure (more accurately, this is what occurs if only one component is present).

2.2.8. *Continuous one-dimensional strong Markov processes.* The structure of continuous one-dimensional strong Markov processes attracted attention in the wake of Feller's paper [69], where he studied the infinitesimal operators of continuous strong Markov processes and proved that in a certain sense they are second-order pseudodifferential operators. However, Feller's papers were purely analytic. In [17] Dynkin established similar results using purely probabilistic methods and characteristic operators instead of infinitesimal operators.

2.3. Martin boundaries and extensions of a Markov process. In the early 1960s Dynkin turned his attention to Martin boundaries. As long ago as the late 1940s Doob was the first to note that Martin boundaries were important for the theory of Markov processes. Hunt later showed that analytic results related to Martin boundaries can also be obtained using purely probabilistic methods. Dynkin first addressed problems involving Martin boundaries at the beginning of the 1960s: in his invited talk[9] at the International Congress of Mathematicians in Stockholm [18] he described, in particular, the limit behaviour of a Brownian motion in the space of ellipsoids, that is, he calculated the corresponding Martin boundary (full proofs are in his subsequent paper [21]). There he also stated a conjecture on the connection between the Martin boundary for a Riemannian manifold and its curvature. Yushkevich noted in [100] that this conjecture was verified in the 1980s in the work of M. T. Anderson, D. Sullivan, R.Schoen, and others. Dynkin's joint paper [16] with Malyutov was related to the same area: there they investigated the properties of the Martin boundary for a random walk on a non-Abelian group with finitely many generators. As mentioned in [100], these results were later used by H. Furstenberg and others.

[9] Delivered by Kolmogorov.

Several of Dynkin's papers were devoted to the so-called oblique derivative boundary-value problem for the Laplace equation (he summarized these investigations in [20]). The analytic statement of the problem is as follows. Let D be a bounded planar domain with smooth boundary and v a vector field on its boundary ∂D. The problem is to find all harmonic functions h in D such that $\partial h/\partial v = 0$ on the boundary. If v is allowed to be tangent to the boundary at some points, then this problem is non-trivial. From the standpoint of Markov processes the problem concerns the behaviour of a Brownian motion in a bounded domain with smooth boundary when there is reflection on the boundary in the direction of v. This problem has been addresses by many authors, starting with Poincaré. A.-M. Liénard proposed a simple method for describing all the solutions of the problem which are smooth up to the boundary. Under the assumption that the field is tangent to the boundary at a finite number of points, Dynkin succeeded in describing all non-negative solutions of the problem that have singularities of any kind at points where the field is tangential.

2.3.1. *Boundary conditions for Markov processes.* The problem of describing all the Markov processes with prescribed local characteristics (or in general with one and the same characteristic operator) borders on this area. Doob was interested in some form of these questions as long ago as the 1940s. Since the characteristic operator describes the behaviour of the process until it goes to infinity or arrives at the boundary of the domain, or in the general case at the Martin boundary, the problem is to find additional boundary conditions which enable one to uniquely characterize the process. In particular, Wentzell [94] investigated general boundary conditions for multidimensional diffusion processes in a domain with smooth boundary. Feller [70] considered a similar problem for Markov processes with a discrete state space, but he could only describe processes subject to an additional condition. Dynkin [23] managed to give a complete answer here, without any additional restrictions. In [22] he considered a similar problem for diffusion processes with an oblique derivative.

2.3.2. *Wanderings of a Markov process.* Let X_t be a standard Markov process in the space E and let D be an open subset of E. The set of times such that $X_t \in D$ is a union of intervals open on the left. The corresponding pieces of the sample path are called wanderings. This concept generalizes the concept of an excursion considered previously by K. Itô. In [31] (see also [25] and [26]) Dynkin investigated wanderings of a Markov process and derived the so-called wandering formula for the expectation of a certain sum of functionals of the wanderings of the process. This formula was later independently proved by R. K. Getoor, M. J. Sharpe, and B. Maisonneuve and was subsequently used under the name 'last exit decomposition'.

2.4. Diffusion in a space of tensors. In developing ideas of Itô [74], Dynkin [24] considered diffusions in a space of tensors and the related notion of stochastic parallel displacement. As noted in [100], this approach was extended to semimartingales in the 1980s by L. Schwartz, Meyer, and I. Shigekawa and has been widely used in stochastic differential geometry.

2.5. Optimal stopping of Markov processes. We mention several papers related to the optimal stopping problem. Dynkin addressed this topic several times.

In [19] he gave a simple and elegant solution of this problem in terms of an excessive majorant of the objective function. His paper [29] treated the game version of the optimal stopping problem. Recently, the approach described in that paper has found extensive applications to mathematical economics under the name of Dynkin games. The reader can find surveys of other works related to mathematical economics in one way or another in [35] and [68].

2.6. Boundary behaviour of Markov processes. Inhomogeneous Markov processes in general state spaces. At the end of the 1960s Dynkin concentrated on the general boundary theory for Markov processes.

2.6.1. *Decompositions of excessive functions into extreme excessive functions.* It is known that a non-negative harmonic function in a domain can be uniquely represented as an integral of the Martin kernel (or the Poisson kernel if the boundary is smooth). This can also be interpreted as the existence of a unique decomposition of non-negative harmonic functions into extreme such functions.

In the general theory of Markov processes the role of non-negative subharmonic functions is played by excessive functions, so it is natural to ask about assumptions ensuring that this result also holds for excessive functions. The answer is closely related to such notions as Martin boundaries and the final behaviour of a Markov process.

The first significant progress here was made by Hunt in [73]. In that lengthy paper he developed a probabilistic analogue of classical potential theory under the assumption that the original process is a right-continuous strong Markov process (probability theory becomes classical potential theory if the model process is a Brownian motion).

Dynkin became interested in all these questions in the second half of the 1960s. He published several papers on the subject, including the two long papers [27] and [28], the first of which treated Markov chains with continuous time and the second of which dealt with general Markov processes. Nevertheless, he was not satisfied with his results: some of his assumptions were clearly of a technical nature.

After several attempts to get rid of the technical restrictions, he changed his approach drastically [30], [32]. He started by looking at inhomogeneous Markov processes and proved the existence of a unique decomposition of inhomogeneous excessive measures and inhomogeneous excessive functions into extreme such measures and functions. In the case of homogeneous processes he was able without great trouble to obtain a decomposition of homogeneous measures and functions into extreme homogeneous measures and functions. Next, he first established a decomposition into extreme measures for excessive measures and then reduced excessive functions to coexcessive measures, which are excessive measures for the dual semigroup. It turns out that this eliminates virtually all restrictions on the properties of sample paths of the process; one does not even need a topology in the state space. Moreover, by making a further small step S. E. Kuznetsov [80] managed to derive from this necessary and sufficient conditions for a decomposition of excessive functions into extreme such functions.

This remarkable gain in generality was obtained not just by changing the strategy, but also thanks to several fundamental ideas and technical results. First, Dynkin connected excessive measures with the one-dimensional distributions of a Markov process with a given transition function. To do this he required a new object, Markov processes with random birth and death times α and β. Such processes are constructed from their two-dimensional distributions as measures in the space of sample paths with a random lifespan. Dynkin carried out this construction in the case when the result is a probability measure, and Kuznetsov later extended it to the case of σ-finite measures [79]. Next, using the simple fact that the transition function of a Markov process along its sample paths is a non-negative martingale (with birth time α), Dynkin constructed the input space for the Markov process in terms of the right-hand limits of such martingales at α and established a one-to-one correspondence between the probability measures on the input space and the Markov processes with the given transition function, that is, with the excessive measures. In [36] he later linked this result with his old paper [6] on sufficient statistics. Making the additional assumption that there exists a transition density and therefore a cotransition function (which is the same as a dual transition function), he reduced the decomposition into extreme excessive functions to the (already established) decomposition of excessive measures for the dual transition function into extreme such measures.

2.6.2. *Regular processes.* After Dynkin's successful treatment of excessive functions it was natural for him to try getting rid of topological assumptions also in other parts of the theory. This was the aim of his papers [33] and [34]. In the first of these he constructed a general theory of Markov processes with random birth and death times. Such properties as right continuity of sample paths of the process were replaced by right continuity of the transition function on sample paths of the process (these are so-called regular Markov processes). He proved the strong Markov property for regular processes. In addition, excessive functions are also right continuous on sample paths of the process. In the second paper he started from a system of σ-algebras $\mathscr{F}(s,t)$ of events observable on time intervals (s,t) which has the Markov property and developed a theory symmetric with respect to time reversal. An important tool in the recovery of the symmetry between the past and the future is the so-called cotransition function. This notion was also used in the problem discussed above of decomposing excessive functions into extreme such functions [30], [32]. Treating the transition and cotransition functions simultaneously is equivalent in a certain sense to treating a pair of dual Markov processes.[10]

2.7. Random fields, Dirichlet forms, and self-intersections of sample paths of a Brownian motion. At the end of the 1970s Dynkin turned to an area which was new for him, random fields. Whereas we can define a stochastic process as a family of random variables X_t, where the parameter t is one-dimensional and interpreted as time, a random field is a family of random variables Z_x where x is interpreted as a point in space (or some set E). Depending on the context, E can be a subset of $\mathbb{R}^n$, the space of smooth functions on $\mathbb{R}^n$, the space of finite measures on a subset of $\mathbb{R}^n$, or something else.

Dynkin's first publication on this subject was [37], where he started from a Markov process with state space E, transition density $p(t,x,y)$, and Green's function $g(x,y) = \int_0^\infty p(t,x,y)\,dt$. Under the assumption that the transition density is symmetric, he constructed a Gaussian random field with zero mean φ_μ

[10] Currently the concept of cotransition function is extensively used in ergodic theory: see [95] and [96], for example.

such that

$$\mathsf{E}\varphi_\mu\varphi_\nu = \int_{E\times E} g(x,y)\,\mu(dx)\,\nu(dy).$$

Here μ and ν are measures on E satisfying a certain finiteness condition. We could do without the space of measures if the Green's function is finite everywhere, but this is not so in the most of the interesting cases. Dynkin proved that the random field thus constructed is Markov and then calculated the conditional expectations of the values of the field when its values on a subdomain of E are known. A similar construction is also possible for several Markov processes. The object thus arising is closely connected with so-called Dirichlet forms. Dirichlet forms and Dirichlet spaces go back to [59] by A. Beurling and J. Deny. Originally, they were connected with the Laplace operator and Brownian motion. M. Fukushima, M. Silverstein, and other authors extended the corresponding theory to right-continuous Markov processes with a symmetric transition function (for instance, see [71]). They assumed that the state space had a topology and, moreover, the corresponding Dirichlet space contained sufficiently many continuous functions. In several papers (including [38] and [39]) Dynkin developed an alternative and, in some respects, more general theory of Dirichlet spaces connected with regular Markov processes with a symmetric transition density. He did not need a topology and replaced the class of continuous functions by the class of right-continuous functions along sample paths of the process.

In [40] Dynkin considered a Markov chain with continuous time on a finite or countable set E and linked it to two random fields on E: τ_x, the sojourn time in the state x, and ν_x, the number of visits to the state x (certain assumptions were made ensuring that both fields are finite with probability 1). Using these fields, he investigated the properties of a random field on E when the Hamiltonian of this field is connected in a certain way with the intensity matrix of the chain under consideration.

In [41] he extended this approach to general Markov processes X_t with a symmetric transition density. In this case the sojourn time T_x in the state x is a generalized random field; it can be defined as the integral $T_B = \int_0^\zeta 1_B(X_t)\,dt$, where B is a subset of the state space and ζ is the killing time of the process. An identity connecting the Gaussian random field constructed in [37] and the field T_x is central to [41]. On the one hand, this identity enables one to use Markov processes in the study of non-Gaussian random fields arising in quantum field theory and similar to those in [92]. On the other hand, the techniques of quantum random fields such as Feynman diagrams can be used in the investigation of local times, self-intersections, and multiple points of a Markov process.

Self-intersections of sample paths of a Brownian motion were the subject of a large cycle of Dynkin's papers. We mention here only two, [42] and [43]. The interest in self-intersections of Brownian sample paths revived after K. Symanzik [93] proposed a cloud of Brownian particles interacting only when their sample paths intersect as a model of a quantum field. In [42] Dynkin connected a certain class of measures with random measures concentrated on a Brownian sample path and also with random measures concentrated on the set of multiple points (that is, points visited some fixed number of times) of the Brownian sample path. The object which arises in this way can be interpreted as a generalized random field with values in a set of random measures. Measures on a sample path of a Brownian motion are equivalent in a certain sense to additive functionals, and measures on the set of multiple points exist only in dimension three or lower. In particular, in the planar case this incudes the so-called two-dimensional Brownian loops. In [43] similar objects were first constructed for a random walk which approximates a Brownian motion, and then the corresponding limits were shown to exist.

2.8. Works on superprocesses. Superprocesses are Markov processes taking values in a space of measures which can be interpreted as probabilistic models of the behaviour of a cloud of branching particles. The first processes of this type were constructed by S. Watanabe [99] and D. Dawson [64]. To help the intuition behind this concept we give a simple example. Assume that some number of Brownian particles move independently in the space. At time β, where β is a small parameter, these particle die, each leaving two descendants (or none) after it with probability 1/2. The new particles also move independently in space according to the law of Brownian motion and die at time 2β, leaving again a random number of descendants after them, and so on. Such a system of particles is known to disappear after a finite (random) number of steps.

We shall assume that each particle has mass β. The positions of the particles at time t can be interpreted as a discrete measure X_t^β (the space distribution of mass). It remains to take the limit as $\beta \to 0$. Furthermore, to obtain a non-trivial limit process we must let the initial number of particles tend to infinity in such a way that the initial mass distribution X_0^β approaches a finite measure μ. Then we obtain a stochastic process (X_t, P_μ) with values in the space of finite measures (here X_t is the sample path of the process in the space of measures, and P_μ is the probability measure describing the behaviour of the process starting with the initial mass distribution μ). The resulting process is called a Dawson–Watanabe super-Brownian motion (although Watanabe did not start with a Brownian motion but rather with a discrete Markov chain, and in both [99] and [64] the process was constructed analytically, without using the above limit procedure at all).

It is well known that a Brownian motion is connected with the Laplace operator ($\frac{1}{2}\Delta$ is its infinitesimal operator). In particular, if $D \subset \mathbb{R}^n$ is a bounded smooth domain, $\varphi(x)$ is a continuous function on the boundary ∂D of D, and ξ_t is a Brownian motion in D which is stopped at a random time τ of exit from the domain, then the function $h(x) = P_x h(\xi_\tau)$ is harmonic and equal to φ on ∂D. In other words, $h(x)$ solves the boundary-value problem $\Delta h = 0$ in D with boundary condition $h = \varphi$.

In turn, a super-Brownian motion is connected with the *non-linear* differential equation $\Delta u = u^2$. To state the corresponding result, we return to the above system of branching particles and assume that the particles arriving at the boundary of D are instantly frozen and do not participate in the further evolution of the system. Let X_D^β denote the mass distribution on the boundary. As $\beta \to 0$ the measure X_D^β tends to a limit which is a random finite measure X_D; it can be interpreted as the mass which has settled on the boundary of D. As above, let $\varphi(x)$ be a continuous function on ∂D. We let $\langle h, X_D\rangle$ denote the integral of the function h with respect to the measure X_D, and we look at the function

$$u(x) = -\log P_{\delta_x} e^{-\langle h, X_D\rangle},$$

where P_{δ_x} is the expectation under the condition that the process starts from the unit mass concentrated at x. It turns out that the function $u(x)$ solves the boundary-value problem $\Delta u = u^2$ in D with boundary condition $u = \varphi$.

2.8.1. *Constructing superprocesses.* A large cycle of Dynkin's works on superprocesses was concerned with their construction and a description of their properties. In fact, in the above construction the Brownian motion can be replaced by an arbitrary Markov random process, the lifetime of the particles can be assumed to be random and governed by a suitable additive functional, and the distribution law for the number of descendants can depend on the position in space and time; in addition, the descendants can be distributed over space in some way. Then there arises the question of conditions ensuring a non-trivial limit process and the question of the properties of such processes, and also whether the class of these processes can be independently described in terms of their properties. These and some other questions were treated in the monograph [44]. Dynkin proved there that a superprocess can be constructed if the original process is a right-continuous strong Markov process without discontinuities of the second kind, and the additive functional mentioned above and the characteristics of the branching law satisfy certain finiteness conditions ([44], Chap. 5; unfortunately, the result itself is too cumbersome to be presented here). The resulting class of processes can be characterized as a class of measure-valued Markov processes with the following property: if the initial measure μ is represented as a sum of two measures $\mu_1 + \mu_2$, then the distribution of the process corresponding to the initial state μ coincides with the distribution of the sum of two independent processes corresponding to the initial states μ_1 and μ_2 [45].

Let us look at the simplest situation where the initial process is a diffusion, the branching is local, the branching characteristics are independent of the position in space, and the expectation of the number of descendants is at most 1. Then the resulting process is connected with the non-linear differential equation

$$Lu = \psi(u),$$

where the second-order elliptic operator L describes the diffusion, and the function $\psi(u)$ characterizes the branching. The ensuing class of functions $\psi(u)$ is described by the formula

$$\psi(u) = bu^2 + \int_0^\infty (e^{-\lambda u} - 1 + \lambda u)\, n(d\lambda),$$

in which we easily recognize the Lévy–Khintchine formula from the theory of infinitely divisible distributions. In particular, this class of functions $\psi(u)$ contains the functions $\psi(u) = u^\alpha$ with $1 < \alpha \leqslant 2$.

2.8.2. *Superprocesses, polar sets, and positive solutions of the equation $\Delta u = u^\alpha$ in a domain.* In a large cycle of papers Dynkin considered the properties of positive solutions of the equation $\Delta u = u^\alpha$ for $1 < \alpha \leqslant 2$ in a domain, using the corresponding super-Brownian motion.[11]

We start with a characterization of the class of polar sets. A set Γ is said to be polar for a Markov process ξ_t, if with probability 1 the process ξ_t never visits Γ.

[11]Some of these results, for instance, the characterization of polar sets for superdiffusions, can also be applied to more general superdiffusions.

In connection with a superprocess X_t this definition has to be modified as follows. We look at the range of the superprocess: the smallest closed set R containing the supports of all the measures X_t. A set Γ is said to be polar for X_t if with probability 1 the range R is disjoint from Γ.

The class of polar sets for a Brownian motion can also be characterized as the class of removable singularities for the Laplace equation. Namely, a set Γ is a removable singularity if each bounded harmonic function in the domain $U \setminus \Gamma$, where U is an open neighbourhood of Γ, can be extended by continuity to a function harmonic in the whole of U. Polar sets can also be described as sets with capacity zero.

Dynkin's interest in this question was prompted by the paper [81] by J.-F. Le Gall, who characterized the removable singularities for the equation $\Delta u = u^2$ in a domain as polar sets and, at the same time, as sets with a certain capacity equal to zero.[12] In conjunction with Kuznetsov, Dynkin [46] was able to prove similar results for a super-Brownian motion corresponding to the case $\psi(u) = u^\alpha$ with $1 < \alpha \leqslant 2$. Namely, the corresponding class of polar sets can be characterized as the class of sets with a certain capacity (the so-called Bessel capacity $C_{2,\alpha/(\alpha-1)}$) equal to zero, or as the class of removable singularities for the equation $\Delta u = u^\alpha$. In this case the definition of a removable singularity has to be modified as follows. Instead of bounded harmonic functions we must consider non-negative (but not necessarily bounded) solutions of the equation $\Delta u = u^\alpha$ in the domain $U \setminus \Gamma$. A similar result also holds for boundary sets with a suitable Bessel capacity equal to zero (for a Brownian motion the corresponding result looks much simpler: a set on the boundary of the domain is polar if it has zero measure).

We have already mentioned that a non-negative harmonic function in a smooth domain D can be represented as an integral $h(x) = \int_{\partial D} k(x,y)\,\mu(du)$, where $k(x,y)$ is the Poisson kernel in D, and $\mu(du)$ is a finite measure on ∂D that in a certain sense serves as the weak limit of the solution on the boundary. The analogues of this classical result for the non-linear equation $\Delta u = u^\alpha$ were treated in a large cycle of Dynkin's papers (some of them joint with Kuznetsov), and were summarized in his monographs [52] and [53].

The class of solutions of the non-linear equation has some properties with no analogues in the class of harmonic functions. For example, there exist solutions which are infinite on a significant part of the boundary or even on the whole boundary (a so-called maximum solution, which dominates all the other solutions). The starting point here was another Le Gall paper, [82], where he established a bijective correspondence between positive solutions of $\Delta u = u^2$ in a bounded planar domain with smooth boundary and their boundary traces, that is, pairs (Γ, ν), where $\Gamma \subset \partial D$ is a closed set near which the solution tends rapidly to infinity, and the σ-finite measure ν on the complement $\partial D \setminus \Gamma$ is the boundary value of the solution in the same sense as for harmonic functions. The planar domain did not appear here by chance: in this dimension there can be no polar sets on the boundary (but they exist in dimension three and higher).

[12] Le Gall's works were concerned with the so-called Brownian snake, a stochastic process with values in the space of sample paths. As in the case of a superprocess, a Brownian snake can be regarded as the limit of a branching system of particles. The snake has some advantages over superprocesses, but restricts the class of possible functions $\psi(u)$. In effect, only the function $\psi(u) = u^2$ is natural for a Brownian snake.

The general answer to this question [52], [53] is connected with the concepts of moderate and σ-moderate solutions. A solution u is moderate if it is majorized by a harmonic function. The boundary value of a moderate solution is a finite measure with no mass on polar sets. A solution u is said to be σ-moderate if it can be represented (non-uniquely, of course) as the limit of a monotonically increasing sequence of moderate solutions. Dynkin and Kuznetsov [49] proved that σ-moderate solutions can be described in terms of their *fine* traces on the boundary, that is, pairs (Γ, ν) where the set $\Gamma \subset \partial D$ is closed in the so-called fine topology connected with the equation and where the σ-finite measure ν on the complement $\partial D \setminus \Gamma$ has no mass on polar sets.[13] Moreover, the corresponding solution can be expressed in terms of the corresponding super-Brownian motion. Finally, we can assert that under certain conditions all non-negative solutions are σ-moderate, so that all positive solutions are uniquely determined by their fine traces. The last result was originally obtained by B. Mselati [88] for $\Delta u = u^2$ in any dimension, and then by Dynkin [53] for $\Delta u = u^\alpha$ with $1 < \alpha \leqslant 2$, again in an arbitrary dimension. Equivalent analytic results which also hold for $\alpha > 2$ were established much later by M. Marcus and L. Véron [83].

2.8.3. *Other results.* In conclusion we mention several papers on superprocesses which lie somewhat to the side of the main direction here. In [50] (a joint paper with Kuznetsov) the authors investigated superprocesses corresponding to the equation $Lu = u^\alpha - cu$, where L is a second-order elliptic operator. They were interested in the existence of a non-trivial solution with boundary data equal to zero. It turns out that such a solution exists in a domain D if and only if with positive probability the superprocess in D does not degenerate. This is possible only for $c > 0$. The papers [54]–[56] were concerned with excessive functions and the Martin boundary for superprocesses. In [47] and [48] the authors described linear additive functionals of superprocesses.

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