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APPROXIMATION BY NÖRLUND TYPE MEANS IN THE GRAND LEBESGUE SPACES WITH VARIABLE EXPONENT

In the present paper the approximation of functions by Nörlund type means in the generalized grand Lebesgue spaces with variable exponent is studied.

Keywords: grand variable exponent Lebesgue spaces, modulus of smoothness, Lipschitz classes, trigonometric approximation, Nörlund means.

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§ 1. Introduction and auxiliary results

Let $\mathbb{T}$ denote the interval $[0, 2\pi]$. We denote by $L^p(\mathbb{T})$, $1 \le p < \infty$, the Lebesgue space of all measurable 2π-periodic functions, for which the norm

$$\|f\|_p = \left(\int_{\mathbb{T}} |f(x)|^p \, dx \right)^{1/p} < \infty.$$

Let us denote by $\wp$ the class of Lebesgue measurable functions $p(\cdot)\colon \mathbb{T} \to [0, \infty)$ such that $1 \le p_* := \operatorname*{ess\,inf}_{x\in\mathbb{T}} p(x) \le \operatorname*{ess\,sup}_{x\in\mathbb{T}} p(x) =: p^* < \infty$. The conjugate exponent of $p(x)$ is shown by $p'(x) := \dfrac{p(x)}{p(x)-1}$. For $p \in \wp$, we define a class $L^{p(\cdot)}(\mathbb{T})$ of 2π-periodic measurable functions $f\colon \mathbb{T} \to \mathbb{C}$ satisfying the condition

$$\int_{\mathbb{T}} |f(x)|^{p(x)} \, dx < \infty.$$

This class $L^{p(\cdot)}(\mathbb{T})$ is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} := \inf\left\{ \lambda > 0\colon \int_{\mathbb{T}} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}. \tag{1.1}$$

The class $L^{p(\cdot)}(\mathbb{T})$ with the norm (1.1) is called the *Lebesgue space with variable exponent*. Information on the properties of this space can be found in [11, 27, 28, 44, 45]. We say that a variable exponent $p(x)$ satisfies the *local log-continuity condition*, if there is a positive constant c such that

$$|p(x) - p(y)| \ln\left(\frac{1}{\log |x-y|} \right) \le c, \tag{1.2}$$

for all $x, y \in [0, 2\pi]$, $|x-y| \le \dfrac{1}{2}$, $x \ne y$.

We denote by $\wp^{\log}(\mathbb{T})$ the class of 2π-periodic functions satisfying the condition (1.2). We also define $\wp_0(\mathbb{T}) := \left\{ p(\cdot) \in \wp^{\log}(\mathbb{T})\colon 1 < p_* \right\}$.

Let $\theta \ge 0$ and $p \in \wp_0(\mathbb{T})$. We denote by the *generalized grand Lebesgue space with variable exponent* $L^{p(\cdot),\theta}$ the class of all 2π-periodic measurable functions f such that

$$\|f\|_{p(\cdot),\theta} = \sup_{0<\varepsilon<p_*-1} \varepsilon^{\frac{\theta}{p_*-\varepsilon}} \|f\|_{p(\cdot)-\varepsilon} < \infty.$$

The space $L^{p(\cdot),\theta}$ was introduced in [30]. Note that when p is a constant and $\theta > 0$, these spaces coincide with the grand Lebesgue spaces introduced by Iwaniec and Sbordone in [20] (for $\theta = 1$) and by Greco, Iwaniec and Sbordone in [16] (for $\theta > 1$). If $p \in \wp$, the embeddings

$$L^{p(\cdot)} \subset L^{p(\cdot),\theta} \subset L^{p(\cdot)-\varepsilon}, \qquad 0 < \varepsilon < p_* - 1,$$

hold.

Note [50] that the generalized grand Lebesgue space with variable exponent $L^{p(\cdot),\theta}$ has important applications in different areas of mathematics, physics and mechanics. In particular, the variable exponent Lebesgue spaces have considerable applications in fluid dynamic, especially, for modeling of electrorheological fluids; the grand and generalized grand Lebesgue spaces have been applied in various fields, in particular, in the theory of PDE [21,42,43], they are right spaces for the investigations of some nonlinear equations. There are sufficient investigations, relating the fundamental problems of these spaces in view of potential theory, maximal and singular operator theory, where the analogues of the classical results existing in the classical Lebesgue spaces were studied. The detailed information about these investigations can be found in the monographs [10, 11, 31, 32, 46].

Note that the closure of the space $L^{p(\cdot)}(\mathbb{T})$ in $L^{p(\cdot),\theta}(\mathbb{T})$, $\theta > 0$, does not coincide with $L^{p(\cdot),\theta}(\mathbb{T})$ [30]. We denote this closure by $L_*^{p(\cdot),\theta}(\mathbb{T})$. This space is a subspace of $L^{p(\cdot),\theta}(\mathbb{T})$. According to [28] for the functions belonging to this space

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{\theta}{p_*-1}} \|f\|_{p(\cdot)-\varepsilon} = 0$$

holds.

We suppose that $p(\cdot) \in \wp_0(\mathbb{T})$ and $\theta > 0$. For $f \in L^{p(\cdot),\theta}(\mathbb{T})$, we set [50]

$$(\nu_h f)(x) := \frac{1}{h}\int_0^h f(x+t)\,dt, \qquad 0 < h < \pi, \quad x \in \mathbb{T}.$$

If $p(\cdot) \in \wp_0(\mathbb{T})$, $\theta > 0$, and $f \in L^{p(\cdot),\theta}(\mathbb{T})$, then the shift operator ν_{h_i} is a bounded linear operator on $L^{p(\cdot),\theta}(\mathbb{T})$ [50]:

$$\|\nu_{h_i}(f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \le c_1 \|f\|_{L^{p(\cdot),\theta}(\mathbb{T})}.$$

Let $p \in \wp_0(\mathbb{T})$, $\theta > 0$, and $f \in L^{p(\cdot),\theta}(\mathbb{T})$. The function

$$\Omega_{p(\cdot),\theta}(f,\delta) := \sup_{0<h\le\delta} \|f - (\nu_h f)\|_{L^{p(\cdot),\theta}(\mathbb{T})}, \qquad \delta > 0,$$

is called the *modulus of continuity* of $f \in L^{p(\cdot),\theta}(\mathbb{T})$.

It can easily be shown that $\Omega_{p(\cdot),\theta}(f,\cdot)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0} \Omega_{p(\cdot),\theta}(f,\delta) = 0, \qquad \Omega_{p(\cdot),\theta}(f+g,\delta) \le \Omega_{p(\cdot),\theta}(f,\delta) + \Omega_{p(\cdot),\theta}(g,\delta), \qquad \delta > 0,$$

for $f, g \in L^{p(\cdot),\theta}(\mathbb{T})$. Note that a detailed information about properties of the modulus of continuity $\Omega_{p(\cdot),\theta}(f,\cdot)$ can be found in the paper [50].

We use the constants $c, c_1, c_2, \ldots$ (in general, different in different relations) which depend only on the quantities that are not important for the questions of interest. We also will use the relation $f = O(g)$ which means that $f \le cg$ for a constant c independent of f and g. Let $0 < \alpha \le 1$. The set of functions $f \in L_*^{p(\cdot),\theta}(\mathbb{T})$ such that

$$\Omega_{p(\cdot),\theta}(f,\delta) = O(\delta^\alpha), \qquad \delta > 0,$$

is called the *Lipschitz class* $\mathrm{Lip}(\alpha, p(\cdot), \theta)$.

Let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} Q_k(x, f) \tag{1.3}$$

be the Fourier series of the function $f \in L_1(T)$, where $Q_k(x, f) := (a_k(f)\cos kx + b_k(f)\sin kx)$, $\alpha_k(f)$ and $b_k(f)$ are Fourier coefficients of the function $f \in L_1(\mathbb{T})$. The n-*th partial sum* of the series (1.3) is defined by

$$S_n(x, f) = \sum_{k=0}^{n} Q_k(x, f),$$

where

$$Q_0(x, f) := \frac{a_0}{2}; \qquad Q_k(x, f) := (a_k(f)\cos kx + b_k(f)\sin kx), \qquad k = 1, 2, \ldots .$$

Let $\{p_n\}_0^\infty$ be a sequence of positive real numbers. The sequence $\{p_n\}_0^\infty$ is called *almost monotone decreasing* (*increasing*), denoted by $\{p_n\}_0^\infty \in$ AMDS ($\{p_n\}_0^\infty \in$ AMIS), if there exists a constant c, depending only on the sequence $\{p_n\}_0^\infty$ such that for all $n \geq m$ the following inequality holds:

$$p_n \leq cp_m \quad (p_m \leq cp_n),$$

In the proof of the main result we will use the notations

$$\Delta\beta_n := \beta_n - \beta_{n+1}, \qquad \Delta_m\beta(n, m) := \beta(n, m) - \beta(n, m+1).$$

As in [38] we suppose that $\mathbb{F}$ is an infinite subset of $\mathbb{N}$ and consider $\mathbb{F}$ as a range of strictly increasing sequence of positive integers, say $\mathbb{F} = \{\lambda(n)\}_1^\infty$. Following [5] and [40], the Cesàro submethod C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \quad n = 1, 2, \ldots,$$

where $\{x_k\}$ is a sequence of real or complex numbers. Therefore, the C_λ-method yields a subsequence of the Cesàro method C_1, and hence it is regular for any λ. C_λ is obtained by deleting a set of rows from Cesàro matrix.

We suppose that $\{p_n\}_0^\infty$ is a sequence of positive real numbers. We define the mean of the series (1.3) as

$$N_n^\lambda(x, f) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} S_m(x; f),$$

where $P_{\lambda(n)} := \sum\limits_{m=0}^{\lambda(n)} p_m \neq 0$, $n \geq 0$, $p_{-1} := P_{-1} := 0$. Note that in the case $p_n = 1$, $n \geq 0$, $N_n^\lambda(x, f)$ is equal to the mean

$$\sigma_n^\lambda(x, f) = \frac{1}{\lambda(n)+1} \sum_{m=0}^{\lambda(n)} S_m(x, f).$$

Using [33, 46] we introduce two new classes of numerical sequences.

Let $R_{\lambda(n),k} = \dfrac{1}{(k+1)P_{\lambda(n)}} \sum\limits_{s=\lambda(n)-k}^{\lambda(n)} p_s$. If $(R_{\lambda(n),k}) \in$ AMDS $\big((R_{\lambda(n),k}) \in$ AMIS$\big)$, then it is said that (p_k) is a λ-*almost monotone decreasing* (*increasing*) *upper mean sequence*, briefly

$(p_k) \in \lambda$-AMDUMS $\big((p_k) \in \lambda$-AMIUMS$\big)$. Note that the classes λ-AMDUMS and λ-AMIUMS are generalizations of the classes AMDUMS and AMIUMS respectively. It is clear that if $\lambda(n) = n$, $n = 1, 2, \ldots$, we obtain λ-AMDUMS $=$ AMDUMS and λ-AMIUMS $=$ AMIUMS defined in [48].

The best approximation of $f \in L_*^{p(\cdot),\theta}$ in the class $\prod_n$ of trigonometric polynomials of degree not exceeding n is defined by

$$E_n(f)_{p(\cdot),\theta} := \inf\left\{\|f - T_n\|_{p(\cdot),\theta} \colon T_n \in \prod\nolimits_n\right\}.$$

In the proof of the main result we need the following Lemmas.

Lemma 1.1 (see [50]). *Let $p \in \wp_0(\mathbb{T})$, $\theta > 0$. Then for $f \in \mathrm{Lip}(\alpha, p(\cdot), \theta)$, $0 < \alpha \le 1$, and $n = 1, 2, 3, \ldots$ the following estimate holds:*

$$\|f - S_n(\cdot, f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} = O(n^{-\alpha}).$$

Lemma 1.2 (see [50]). *Let $p \in \wp_0(\mathbb{T})$, $\theta > 0$. Then for $f \in \mathrm{Lip}(1, p(\cdot), \theta)$ and $n = 1, 2, 3, \ldots$ the following estimate holds:*

$$\|S_n(\cdot, f) - \sigma_n(\cdot, f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} = O(n^{-1}).$$

Lemma 1.3 (see [33]). *Let $\{p_n\}$ be a positive sequence. Let following conditions hold:*

(1) *$(p_n) \in \lambda$-AMDUMS or*

(2) *$(p_n) \in \lambda$-AMIUMS and $(\lambda(n) + 1)p_{\lambda(n)} = O(P_{\lambda(n)})$.*

Then

$$\Lambda := \sum_{m=0}^{\lambda(n)} \frac{p_{\lambda(n)-m}}{(m+1)^\alpha} = O_\alpha\left(\frac{P_{\lambda(n)}}{(\lambda(n)+1)^\alpha}\right).$$

for $0 < \alpha < 1$.

Theorem 1.1 (see [33]). *The following properties are valid:*

(1) *if $(p_n) \in$ AMDS, then $(p_n) \in \lambda$-AMIUMS;*

(2) *if $(p_n) \in$ AMIS, then $(p_n) \in \lambda$-AMDUMS;*

(3) *if $\sum\limits_{s=0}^{\lambda(n)-1} \left|\Delta\left(\frac{p_s}{P_{\lambda(n)}}\right)\right| = O\left((\lambda(n))^{-1}\right)$, then $\sum\limits_{s=0}^{\lambda(n)-1} \left|\Delta(R_{\lambda(n),s})\right| = O\left((\lambda(n))^{-1}\right)$;*

(4) *if $\sum\limits_{s=1}^{\lambda(n)-1} s\left|\Delta\left(\frac{p_s}{P_{\lambda(n)}}\right)\right| = O(1)$, then $\sum\limits_{s=0}^{\lambda(n)-2} \left|\Delta(R_{\lambda(n),s})\right| = O\left((\lambda(n))^{-1}\right)$.*

§ 2. Main Results

The problems of approximation theory in variable and grand variable exponent Lebesgue spaces have been investigated by several authors (see, for example, [1–4, 12–15, 17, 22, 26, 29, 47, 50–52, 54, 55]). In the present paper we study the approximation of functions by Nörlund type means in the generalized grand Lebesgue space with variable exponent $L^{p(\cdot),\theta}$, $\theta > 0$. The results obtained in this work are generalization of the results [17, 32] to the generalized grand Lebesgue space with variable exponent. Similar approximation problems in different spaces have been investigated in [6–9, 9, 17–19, 22–26, 33–41, 48–54, 56–58].

Note that, in the proof of the main results we use the method as in the proofs of [17, 33].

Our main results are the following.

Theorem 2.1. *Let* $p \in \wp_0(\mathbb{T})$, $\theta > 0$, *and* $\{p_n\}_{n=0}^{\infty}$ *be a sequence of positive real numbers. Also, let the following conditions hold:*

$$\{p_n\}_0^{\infty} \in \lambda\text{-}AMDUMS$$

or

$$\{p_n\}_0^{\infty} \in \lambda\text{-}AMIUMS, \text{ and } (\lambda(n)+1)p_{\lambda(n)} = O(P_{\lambda(n)}). \tag{2.1}$$

Then, for $f \in \mathrm{Lip}(\alpha, p(\cdot), \theta)$, $0 < \alpha < 1$, *the relation*

$$\|f - N_n^{\lambda}(\cdot, f)\|_{L^{p(\cdot,\theta)}(\mathbb{T})} = O\left((\lambda(n)+1)^{-\alpha}\right), \quad n \in \mathbb{N} \cup \{0\},$$

holds.

Theorem 2.2. *Let* $p \in \wp_0(\mathbb{T})$, $\theta > 0$, *and* $\{p_n\}_{n=0}^{\infty}$ *be a sequence of positive real numbers. Also, let the following condition holds:*

$$\sum_{m=0}^{\lambda(n)-2} |R_{\lambda(n),m} - R_{\lambda(n),m+1}| = O\left((\lambda(n))^{-1}\right).$$

Then, for $f \in \mathrm{Lip}(1, p(\cdot), \theta)$, *the relation*

$$\|f - N_n^{\lambda}(\cdot, f)\| L^{p(\cdot,\theta)}(\mathbb{T}) = O\left((\lambda(n))^{-1}\right), \quad n = 1, 2, \ldots .$$

holds.

Remark 2.1. Theorem 2.2 gives the same degree of approximation with conditions different from those of Theorem 1.1, considering the case $\alpha = 1$.

Remark 2.2. If $\theta = 0$ and $\lambda(n) = n$, $n = 1, 2, \ldots$, then from Theorems 2.1 and 2.2 we obtain results of [17].

Note that, since $(\lambda(n))^{-\alpha} \le n^{-\alpha}$, $0 < \alpha \le 1$, the results obtained in [33] give sharper estimates than those of results in [17].

§ 3. Proofs of the main results

Proof of Theorem 2.1. It is clear that

$$N_n^{\lambda}(x, f) - f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m}\{f(x) - S_m(x, f)\}. \tag{3.1}$$

Then using Lemma 1.1 and Lemma 1.3 and (2.1) we have

$$\begin{aligned}
\|N_n^{\lambda}(\cdot, f) - f\|_{L^{p(\cdot),\theta}(\mathbb{T})} &\le \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m}\|f - S_m(\cdot, f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \\
&= \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m}\|f - S_m(\cdot, f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} + \|f - S_0(\cdot, f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \\
&= \frac{1}{P_{\lambda(n)}} O\left(\sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m}(m+1)^{-\alpha}\right) \\
&= \frac{1}{P_{\lambda(n)}} O\left(P_{\lambda(n)}(\lambda(n)+1)^{-\alpha}\right) = O\left((\lambda(n)+1)^{-\alpha}\right).
\end{aligned}$$

The proof of the Theorem is completed. $\square$

Proof of Theorem 2.2. We can write the following equality:

$$F_n^{\lambda}(x,f) := N_n^{\lambda}(x,f) - f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m}\{S_m(x,f) - f(x)\}.$$

Using Abel's transformation, we find that [33]

$$\begin{aligned} F_n^{\lambda}(x,f) &= \sum_{m=1}^{\lambda(n)-1} \big(S_m(x,f) - S_{m+1}(x,f)\big) \frac{1}{P_{\lambda(n)}} \sum_{s=0}^{m} p_{\lambda(n)-s} + S_{\lambda(n)}(x,f) - f(x) \\ &= -\sum_{m=0}^{\lambda(n)-1} (m+1) Q_{m+1}(x,f) R_{\lambda(n),m} + S_{\lambda(n)}(x,f) - f(x) \\ &= -\sum_{m=0}^{\lambda(n)-2} (R_{\lambda(n),m} - R_{\lambda(n),m+1}) \sum_{s=0}^{m} (s+1) Q_{s+1}(x,f) \\ &\quad - \big((\lambda(n) P_{\lambda(n)})^{-1}\big) \sum_{s=1}^{\lambda(n)} p_s \sum_{s=0}^{\lambda(n)-1} (s+1) Q_{s+1}(x,f) + S_{\lambda(n)}(x,f) - f(x) \end{aligned} \tag{3.2}$$

Using (3.2), we obtain

$$\begin{aligned} \|F_n^{\lambda}(\cdot,f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} \leq & \sum_{m=0}^{\lambda(n)-2} |R_{\lambda(n),m} - R_{\lambda(n),m+1}| \left\| \sum_{s=1}^{m-1} s Q_s(\cdot,f) \right\|_{L^{p(\cdot),\theta}(\mathbb{T})} \\ & + \big((\lambda(n))^{-1}\big) \left\| \sum_{s=1}^{\lambda(n)} s Q_s(\cdot,f) \right\|_{L^{p(\cdot),\theta}(\mathbb{T})} + \|S_{\lambda(n)}(\cdot,f) - f\|_{L^{p(\cdot),\theta}(\mathbb{T})}. \end{aligned}$$

It is clear that the equality

$$\sum_{s=1}^{\lambda(n)} s Q_s(f;x) = \big(\lambda(n)+1\big) S_{\lambda(n)}(x,f) - \sigma_{\lambda(n)}(x,f)$$

$$S_n(x,f) - \sigma_n(x,f) = \frac{1}{n+1} \sum_{k=1}^{n} k Q_k(x,f). \tag{3.3}$$

holds. Then, from Lemma 1.2 and (3.3), we have

$$\left\| \sum_{s=1}^{\lambda(n)} s Q_s(\cdot,f) \right\|_{L^{p(\cdot),\theta}(\mathbb{T})} = O(1). \tag{3.4}$$

Thus, use of Lemma 1.2 and (3.4) gives us

$$\|F_n^{\lambda}(\cdot,f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} = O\left(\sum_{m=0}^{\lambda(n)-2} |R_{\lambda(n),m} - R_{\lambda(n),m+1}| \right) + O((\lambda(n))^{-1}). \tag{3.5}$$

Now we suppose that the condition

$$\sum_{m=0}^{\lambda(n)-2} |R_{\lambda(n),m} - R_{\lambda(n),m+1}| = O\left(\lambda(n)^{-1}\right)$$

is satisfied. Then the last relation and (3.5) imply that

$$\|f - N_n^\lambda(\cdot, f)\|_{L^{p(\cdot),\theta}(\mathbb{T})} = O\left((\lambda(n))^{-1}\right), \quad n = 1, 2, \ldots.$$

The proof of Theorem 2.2 is completed. □

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Аппроксимация средними Нёрлунда в гранд-пространствах Лебега с переменным показателем

Ключевые слова: гранд-пространства Лебега с переменным показателем, модуль гладкости, классы Липшица, тригонометрическая аппроксимация, средние Нёрлунда.

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В настоящей работе исследуется аппроксимация функций средними Нёрлунда в обобщенных гранд-пространствах Лебега с переменным показателем.

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