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Estimates for Restrictions of Monotone Operators on the Cone of Decreasing Functions in Orlicz Space

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Abstract—The restriction of a monotone operator P to the cone Ω of nonnegative decreasing functions from a weighted Orlicz space $L_{\varphi,v}$ without additional a priori assumptions on the properties of the Orlicz function φ and the weight function v is considered. An order-sharp two-sided estimate of the norm of this restriction is established by using a specially constructed discretization procedure. Similar estimates are also obtained for monotone operators over the corresponding Orlicz–Lorentz spaces $\Lambda_{\varphi,v}$. As applications, descriptions of associated spaces for the cone Ω and the Orlicz–Lorentz space are obtained. These new results are of current interest in the theory of such spaces.

Key words: *monotone operator*, *weighted Orlicz space*, *cone of decreasing functions*, *associated norm*, *Orlicz–Lorentz class*, *discretization method.*

1. DEFINITIONS AND PRELIMINARY RESULTS

Definition 1. Let A denote the class of functions φ possessing the following properties: the inclusion $\varphi \in A$ means that φ belongs to $C[0,\infty)$, $\varphi(0)=0$; φ is a strictly increasing function and, for any $c \in \mathbb{R}_+ = (0,\infty)$,

$$d(c) := \inf\{d>0 : \varphi(dt) \ge c\varphi(t) \text{ for all } t \in \mathbb{R}_+\} < \infty. \tag{1}$$

Obviously, for any function $\varphi\uparrow$, we have

$$c \in (0,1] \quad\Longrightarrow\quad d(c) \in (0,1], \qquad c>1 \quad\Longrightarrow\quad d(c)\ge 1. \tag{2}$$

In addition, the limit $\lim_{t\to+0}[t^{-1}\varphi(t)]$ exists and

$$\lim_{t\to+0}[t^{-1}\varphi(t)] \in \mathbb{R}_+ \quad\Longrightarrow\quad d(c)\ge c. \tag{3}$$

Example 1. If $\varphi(t)=t^{\varepsilon}$, $t\in[0,\infty)$, $\varepsilon>0$, then $d(c)=c^{1/\varepsilon}$.

Example 2. Let $\varphi(t)=e^t-1$. Then

$$c \in (0,1] \quad\Longrightarrow\quad d(c)=1, \qquad c>1 \quad\Longrightarrow\quad d(c)=c. \tag{4}$$

Example 3. Let

$$\varphi(t)=\ln^{\gamma}(t+1), \qquad t\in[0,\infty), \quad \gamma>0.$$

Then $d(c) = \infty$ for any $c > 1$. Indeed, if $c > 1$, then, for any $d \in \mathbb{R}_+$, the inequality

$$\ln^\gamma(dt + 1) \geq c \ln^\gamma(t + 1)$$

fails for a sufficiently large t, $t \in \mathbb{R}_+$, because

$$\lim_{t \to +\infty} \left[\frac{\ln^\gamma(dt + 1)}{\ln^\gamma(t + 1)} \right] = 1. \tag{5}$$

Example 4. Let $\varepsilon > 0$, and let $\varphi(t)t^{-\varepsilon} \uparrow$ on $\mathbb{R}_+$. Then

$$d(c) \leq \max\{1, c^{1/\varepsilon}\}. \tag{6}$$

Indeed, for $c \in (0, 1]$, we have $d(c) \in (0, 1]$; see (2). If $c > 1$, then

$$\varphi(c^{1/\varepsilon}t) = (c^{1/\varepsilon}t)^\varepsilon [\varphi(c^{1/\varepsilon}t)(c^{1/\varepsilon}t)^{-\varepsilon}] \geq (c^{1/\varepsilon}t)^\varepsilon [\varphi(t)t^{-\varepsilon}] = c\varphi(t)$$

for any $t \in \mathbb{R}_+$. Thus, $d(c) \leq c^{1/\varepsilon}$ for $c > 1$.

Remark 1. For the general properties of functions in Orlicz spaces, see [1] and [2]. Note that

$$\varphi \in A \quad \Longrightarrow \quad \varphi(+\infty) = \infty,$$

so that φ is the so-called φ-function; see [1, Chap. 2]. At the same time, each Orlicz function (i.e., a convex φ-function; see [1, Chap. 10]), belongs to A. Indeed, the Orlicz function φ is convex, so that $\varphi(t)t^{-1} \uparrow$, and Example 4 with $\varepsilon = 1$ includes such functions. Note that, for Orlicz functions,

$$d(c) \leq \max\{1, c\}.$$

This is also valid for N functions (see [1], [2, Chap. 8] as well as Sec. 4 below). Let $M = M(R_+)$ denote the set Lebesgue measurable functions on $\mathbb{R}_+$, and let $M^+ = \{f \in M : f \geq 0\}$. Without loss of generality, we shall assume in what follows that all measurable functions are finite almost everywhere with respect to Lebesgue measure on $\mathbb{R}_+$.

For $f \in M$, we define the functional

$$J_\lambda(f) := \int_0^\infty \varphi(\lambda^{-1}|f(t)|)v(t)\, dt, \qquad \lambda > 0 \tag{7}$$

(here integration is over Lebesgue measure). Everywhere in what follows, we shall assume that the weight function $v \in M^+$ satisfies the conditions $0 < v < \infty$ almost everywhere. In addition, beginning with the second part of Sec. 2, we assume that

$$0 < V(t) := \int_0^t v\, d\tau < \infty \qquad \text{for all} \quad t \in \mathbb{R}_+. \tag{8}$$

In addition, we assume that V is a strictly increasing function and

$$V(+\infty) = \infty. \tag{9}$$

Now, for $f \in M$, let us define the Luxemburg functional

$$\|f\|_{\varphi,v} := \inf\{\lambda > 0 : J_\lambda(f) \leq 1\}. \tag{10}$$

Then, for $c \in [0, \infty)$, $f, g \in M$, we have

$$\|cf\|_{\varphi,v} = c\|f\|_{\varphi,v}, \qquad |f| \leq |g| \quad \Longrightarrow \quad \|f\|_{\varphi,v} \leq \|g\|_{\varphi,v}.$$

Consider a weighted Orlicz space $L_{\varphi,v}$ containing the cone of nonnegative functions:

$$L_{\varphi,v} = \{f \in M : \|f\|_{\varphi,v} < \infty\}, \qquad L^+_{\varphi,v} = \{f \in L_{\varphi,v} : f \geq 0\}. \tag{11}$$

The following propositions will make it possible to evaluate $\|f\|_{\varphi,v}$ using the properties of $J_\lambda(f)$. The first of these shows that a two-sided estimate of $J_\lambda(f)$ implies the corresponding two-sided estimate for $\|f\|_{\varphi,v}$.

Proposition 1. *Suppose that $\varphi \in A$, $c \in \mathbb{R}_+$, and $f_1, f_2 \in M$. If, for all $\lambda > 0$,*

$$J_\lambda(f_1) \leq cJ_\lambda(f_2), \tag{12}$$

then

$$\|f_1\|_{\varphi,v} \leq d(c)\|f_2\|_{\varphi,v}, \tag{13}$$

where $d(c)$ is the constant defined in (1).

Corollary 1. *In particular, we have*

$$J_\lambda(f_1) \leq J_\lambda(f_2) \quad \textit{for all} \quad \lambda > 0 \qquad \Longrightarrow \qquad \|f_1\|_{\varphi,v} \leq \|f_2\|_{\varphi,v}. \tag{14}$$

Indeed, if $c = 1$ in (12), then $d(c) \leq 1$ in (13); see (2).

Corollary 2. *Suppose that $0 < c_1 \leq c_2 < \infty$ and $f_1, f_2 \in M$. If, for any $\lambda > 0$,*

$$c_1 J_\lambda(f_2) \leq J_\lambda(f_1) \leq c_2 J_\lambda(f_2), \tag{15}$$

then

$$d_1\|f_2\|_{\varphi,v} \leq \|f_1\|_{\varphi,v} \leq d_2\|f\|_{\varphi,v}, \tag{16}$$

where (*see* (1))

$$d_1 = d(c_1^{-1})^{-1}, \qquad d_2 = d(c_2). \tag{17}$$

Proof of Proposition 1. It follows from (12) that

$$\|f_1\|_{\varphi,v} = \inf\{\lambda > 0 : J_\lambda(f_1) \leq 1\} \leq \inf\{\lambda > 0 : cJ_\lambda(f_2) \leq 1\}. \tag{18}$$

If $c \in (0, 1]$, then this inequality implies estimate (13) with $d(c) = 1$.

Now let $c > 1$. Then, for $d > d(c)$, we have $c\varphi(t) \leq \varphi(dt)$ for all $t \in \mathbb{R}_+$, and thus,

$$cJ_\lambda(f_2) \leq J_\lambda(df_2).$$

Therefore,

$$\inf\{\lambda > 0 : cJ_\lambda(f_2) \leq 1\} \leq \inf\{\lambda > 0 : J_\lambda(df_2) \leq 1\} = \|df_2\|_{\varphi,v}. \tag{19}$$

From (18) and (19), we obtain

$$\|f_1\|_{\varphi,v} \leq \|df_2\|_{\varphi,v} = d\|f_2\|_{\varphi,v}. \tag{20}$$

Inequality (20) holds for all $d > d(c)$. Thus, (20) implies (13). $\square$

The following well-known results can be used to calculate the norm of an operator on a weighted Orlicz space (for known results on interpolation and for general properties of Orlicz spaces, see the statements and references in [1]–[6]).

Proposition 2. *Let $\varphi \in C[0, \infty)$, $\varphi(0) = 0$, be an increasing function. Then, for $f \in M$, $v \in M^+$, the following equivalence holds*:

$$\|f\|_{\varphi,v} \leq 1 \quad \Longleftrightarrow \quad J_1(f) = \int_0^\infty \varphi(|f(t)|)v(t)\,dt \leq 1. \tag{21}$$

Proposition 3. *Suppose that φ be p-convex for some $p \in (0,1]$, i.e., for $\alpha, \beta \geq 0$, $\alpha^p + \beta^p = 1$,*

$$\varphi(\alpha t + \beta\tau) \leq \alpha^p \varphi(t) + \beta^p \varphi(\tau), \qquad t, \tau \in [0, \infty). \tag{22}$$

Then, for the space $L_{\varphi,v}$, the following triangle inequality holds:

$$\|f + g\|_{\varphi,v} \leq (\|f\|^p_{\varphi,v} + \|g\|^p_{\varphi,v})^{1/p}. \tag{23}$$

Remark 2. Note that if φ, is p-convex, then $\varphi(t)t^{-p}$ is an increasing function, so that $\varphi \in A$; see Example 4. Thus, in this case, all the results given above can be used.

Remark 3. Let φ, be p-convex for some $p \in (0,1]$, and let the weight function v satisfy the condition $0 < v < \infty$ almost everywhere. Then $L_{\varphi,v}$ is an ideal space, i.e., the quasi-Banach space (Banach space for $p = 1$) of measurable functions equipped with monotone quasinorm $\|\cdot\|_{\varphi,v}$, (norm for $p = 1$). In addition, this space possesses the Fatou property

$$f_m \in M^+, \qquad f_m \uparrow f \quad \mu\text{-a.e.} \quad \Longrightarrow \quad \|f_m\|_{\varphi,v} \uparrow \|f\|_{\varphi,v}. \tag{24}$$

Remark 4. The general theory of monotone operators and cones in Banach lattices was presented in [7]–[10], and results concerning operators in ideal and rearrangement-invariant spaces were given in [3], [6], as well as in [11]. Duality in Orlicz spaces, Lorentz, and Orlicz–Lorentz spaces, including the descriptions of associated norms on various versions of these spaces, were studied, in particular, in [2], [4], [12]–[16]. Many studies in this direction were based on the Sawyer's fundamental results [17].

2. DISCRETIZATION PROCEDURE

1. First, let us briefly describe the discrete version of the constructions given above. Let $\varphi \in A$; see Definition 1. Suppose we are given a weighted sequence

$$\beta = \{\beta_m\}, \quad \beta_m \in \mathbb{R}_+ = (0, \infty), \qquad m \in \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}.$$

By analogy with (7)–(11), for $\alpha = \{\alpha_m\}$, $\alpha_m \in \mathbb{R}$, we define

$$j_\lambda(\alpha) := \sum_m \varphi(\lambda^{-1}|\alpha_m|)\beta_m, \qquad \|\alpha\|_{l_{\varphi,\beta}} := \inf\{\lambda > 0 : j_\lambda(\alpha) \leq 1\}. \tag{25}$$

The following formulas are valid:

$$j_\lambda(0 \cdot \alpha) = 0, \qquad c \in \mathbb{R}_+ \quad \Longrightarrow \quad j_\lambda(c\alpha) = j_{\lambda/c}(\alpha). \tag{26}$$

Also let $\alpha = \{\alpha_m\}$, $\gamma = \{\gamma_m\}$. Then

$$|\alpha_m| \leq |\gamma_m| \quad \Longrightarrow \quad j_\lambda(\alpha) \leq j_\lambda(\gamma) \quad \Longrightarrow \quad \|\alpha\|_{l_{\varphi,\beta}} \leq \|\gamma\|_{l_{\varphi,\beta}}, \tag{27}$$

$$\|c\alpha\|_{l_{\varphi,\beta}} = c\|\alpha\|_{l_{\varphi,\beta}}, \qquad c \in \mathbb{R}_+. \tag{28}$$

The following analogs of Propositions 1 and 2 are valid.

Proposition 4. *Suppose that $\varphi \in A$, $c \in \mathbb{R}_+$ and $\alpha = \{\alpha_m\}$, $\gamma = \{\gamma_m\}$. If, for any $\lambda > 0$,*

$$j_\lambda(\alpha) \leq c j_\lambda(\gamma), \tag{29}$$

then

$$\|\alpha\|_{l_{\varphi,\beta}} \leq d(c)\|\gamma\|_{l_{\varphi,\beta}} \tag{30}$$

where $d(c)$ is the constant defined in (1).

Corollary 3. *In particular, we have*

$$j_\lambda(\alpha) \le j_\lambda(\gamma) \quad \textit{for all} \quad \lambda > 0 \qquad \Longrightarrow \qquad \|\alpha\|_{l_{\varphi,\beta}} \le \|\gamma\|_{l_{\varphi,\beta}}. \tag{31}$$

Corollary 4. *Suppose that* $0 < c_1 \le c_2 < \infty$ *and* $\alpha = \{\alpha_m\}$, $\gamma = \{\gamma_m\}$. *If, for any* $\lambda > 0$,

$$c_1 j_\lambda(\gamma) \le j_\lambda(\alpha) \le c_2 j_\lambda(\gamma), \tag{32}$$

then, for $d_1 = d(c_1^{-1})^{-1}$, $d_2 = d(c_2)$ *(see* (1)*), the following estimates hold*:

$$d_1\|\gamma\|_{l_{\varphi,\beta}} \le \|\alpha\|_{l_{\varphi,\beta}} \le d_2\|\gamma\|_{l_{\varphi,\beta}}. \tag{33}$$

Proposition 5. *Let* $\varphi \in C[0,\infty)$, $\varphi(0) = 0$, *be an increasing function. Then, for* $\alpha = \{\alpha_m\}$, $\beta = \{\beta_m\}$, $\beta_m \in \mathbb{R}_+$, *the following equivalence holds*:

$$\|\alpha\|_{l_{\varphi,\beta}} \le 1 \quad \Longleftrightarrow \quad j_1(\alpha) = \sum_m \varphi(|\alpha_m|)\beta_m \le 1. \tag{34}$$

To justify these discrete analogs of the corresponding results from Sec. 1, we can introduce a sequence $\{\mu_m\}$ such that

$$\mu_m < \mu_{m+1}, \qquad R_+ = \bigcup_m \Delta_m, \qquad \Delta_m = [\mu_m, \mu_{m+1}). \tag{35}$$

We introduce the weight function $v > 0$ satisfying the conditions

$$\int_{\Delta_m} v\,dt = \beta_m. \tag{36}$$

Further, let us narrow down the required results from Sec. 1 to the set of step-functions

$$\widetilde{L}_{\varphi,v} = \Big\{ f \in L_{\varphi,v} : f = \sum_m \alpha_m \chi_{\Delta_m},\ \alpha_m \in \mathbb{R} \Big\}, \tag{37}$$

where χ_{Δ_m} is the characteristic function of the interval Δ_m. For such functions,

$$J_\lambda(f) = j_\lambda(\alpha), \qquad \|f\|_{\varphi,v} = \|\alpha\|_{l_{\varphi,\beta}}. \tag{38}$$

Indeed,

$$J_\lambda(f) = \int_0^\infty \varphi(\lambda^{-1}|f(t)|)v(t)\,dt = \sum_m \int_{\Delta_m} \cdots = \sum_m \varphi(\lambda^{-1}|\alpha_m|) \int_{\Delta_m} v\,dt = j_\lambda(\alpha).$$

Now all the discrete formulas mentioned above are particular cases of the corresponding formulas from Sec. 1.

2. Let us describe the discretization procedure for integral relations on the cone of decreasing functions in the spaces $L_{\varphi,v}$:

$$\Omega \equiv \{f \in L_{\varphi,v} : 0 \le f \downarrow\}. \tag{39}$$

We assume that the weight function v satisfies conditions (8) and (9). For a fixed number $b > 1$, we introduce the sequence $\{\mu_m\}$ by the formulas

$$\mu_m = V^{-1}(b^m) \quad \Longleftrightarrow \quad V(\mu_m) = b^m, \qquad m \in \mathbb{Z} = \{0, \pm1, \pm2, \dots\}, \tag{40}$$

where V^{-1} is the inverse function to the continuous increasing function V. Then conditions (35) hold, because

$$0 < \mu_m \uparrow, \qquad \lim_{m\to-\infty} \mu_m = 0, \qquad \lim_{m\to+\infty} \mu_m = \infty.$$

Let us also introduce the cones of nonnegative step-functions

$$S \equiv L^+_{\varphi,v} \cap \widetilde{L}_{\varphi,v} = \Big\{ f \in L_{\varphi,v} : f = \sum_m \gamma_m \chi_{\Delta_m},\ \gamma_m \ge 0,\ m \in \mathbb{Z} \Big\}, \tag{41}$$

and of nonnegative decreasing step-functions

$$\widetilde{\Omega} \equiv \Omega \cap \widetilde{L}_{\varphi,v} = \Big\{ f \in L_{\varphi,v} : f = \sum_m \alpha_m \chi_{\Delta_m},\ 0 \le \alpha_m \downarrow \Big\}. \tag{42}$$

For $f \in \Omega$, we define the step-functions $f_0, f_1 \in \widetilde{\Omega}$ (see (35)) as

$$f_0 := \sum_m f(\mu_{m+1}) \chi_{\Delta_m}, \qquad f_1 := \sum_m f(\mu_m) \chi_{\Delta_m}. \tag{43}$$

Then

$$f_0 \le f \le f_1 \quad \Longrightarrow \quad \|f_0\|_{\varphi,v} \le \|f\|_{\varphi,v} \le \|f_1\|_{\varphi,v} \tag{44}$$

(the left inequality in (44) holds everywhere on R_+).

Let us apply relations (36)–(38) to the step-functions f_0 and f_1, obtaining

$$\|f_0\|_{\varphi,v} = \|\{\alpha_{m+1}\}\|_{l_{\varphi,\beta}}, \qquad \|f_1\|_{\varphi,v} = \|\{\alpha_m\}\|_{l_{\varphi,\beta}}, \qquad \alpha_m := f(\mu_m). \tag{45}$$

Here, by (36) and (40), we have

$$\beta_m = \int_{\Delta_m} v\, dt = V(\mu_{m+1}) - V(\mu_m) = b^m(b-1), \qquad m \in \mathbb{Z}. \tag{46}$$

Remark 5. For a discretization of the form (40)–(46), the shift operators

$$T_+[\{\gamma_m\}] = \{\gamma_{m+1}\}, \qquad T_-[\{\gamma_m\}] = \{\gamma_{m-1}\} \tag{47}$$

turn out to be bounded as operators in $l_{\varphi,\beta}$.

This is a particular case of the following result.

Lemma 1. *Suppose that $b > 1$, $\varphi \in A$, $\beta = \{\beta_m\}$, $\beta_m \in R_+$, $1 \le \beta_{m+1}/\beta_m \le b$, $m \in \mathbb{Z}$. Then*

$$\|T_+\| \le 1, \qquad \|T_-\| \le d(b), \tag{48}$$

where $d(b)$ is the constant (1) *for $c = b > 1$. If φ is a convex function* (*in particular, if φ is an N-function*), *we obtain estimates* (48) *with $d(b) = b$.*

Proof. To obtain estimates (48), note that

$$j_\lambda(\{\gamma_{m+1}\}) \le j_\lambda(\{\gamma_m\}), \qquad j_\lambda(\{\gamma_{m-1}\}) \le b j_\lambda(\{\gamma_m\}). \tag{49}$$

Indeed,

$$\begin{aligned} j_\lambda(\{\gamma_{m+1}\}) &= \sum_{m\in\mathbb{Z}} \varphi(\lambda^{-1}|\gamma_{m+1}|)\beta_m = \sum_{m\in\mathbb{Z}} \varphi(\lambda^{-1}|\gamma_m|)\beta_{m-1}, \\ j_\lambda(\{\gamma_{m-1}\}) &= \sum_{m\in\mathbb{Z}} \varphi(\lambda^{-1}|\gamma_{m-1}|)\beta_m = \sum_{m\in\mathbb{Z}} \varphi(\lambda^{-1}|\gamma_m|)\beta_{m+1}; \end{aligned}$$

hence we obtain inequalities (49) by taking into account the conditions on $\beta = \{\beta_m\}_{m\in\mathbb{Z}}$. It follows from (49) and (29)–(31) that

$$\begin{aligned} \|T_+[\{\gamma_m\}]\|_{l_{\varphi,\beta}} &= \|\{\gamma_{m+1}\}\|_{l_{\varphi,\beta}} \le \|\{\gamma_m\}\|_{l_{\varphi,\beta}}, \\ \|T_-[\{\gamma_m\}]\|_{l_{\varphi,\beta}} &= \|\{\gamma_{m-1}\}\|_{l_{\varphi,\beta}} \le d(b)\|\{\gamma_m\}\|_{l_{\varphi,\beta}}. \end{aligned} \tag{50}$$

If φ is convex, then $d(b) = \max\{1, b\} = b$. Thus, we obtain estimates (48). $\square$

Let us apply estimate (48) to the sequence $\{\gamma_m\} = \{\alpha_{m+1}\}$. Then, by (45), we have

$$\|f_1\|_{\varphi,v} = \|\{\alpha_m\}\|_{l_{\varphi,\beta}} \le d(b)\|\{\alpha_{m+1}\}\|_{l_{\varphi,\beta}} = d(b)\|f_0\|_{\varphi,v}. \tag{51}$$

Substituting (51) into (44), we obtain the following conclusion.

Remark 6. Suppose that $b > 1$, $\varphi \in A$ and the weight v satisfies conditions (8), (9). Suppose that the discretization procedure (40)–(46) is applied to the function $f \in \Omega$ (see (39)). Then

$$d(b)^{-1}\|f_1\|_{\varphi,v} \le \|f\|_{\varphi,v} \le \|f_1\|_{\varphi,v}, \tag{52}$$

where $d(b)$ is defined in (1) for $c = b > 1$. Here the step-function f_1 defined in (43), satisfies relation (45).

3. ESTIMATES FOR THE NORM OF A MONOTONE OPERATOR ON THE CONE Ω

We shall preserve all the notation from Secs. 1 and 2. Let (N, η) be a space with nonnegative complete σ-finite measure η, let $L = L(N, \eta)$ be the set of all η-measurable functions $u\colon N \to \mathbb{R}$, and let $L^+ = \{u \in L : u \ge 0\}$. Here we assume that the pointwise inequalities hold η-almost everywhere. Let $Y = Y(N, \eta) \subset L$ be an ideal space, i.e., the Banach (or quasi-Banach) space of measurable functions with monotone norm (or quasinorm) $\|\cdot\|_Y$, so that

$$u_1 \in L, \quad |u_1| \le |u_2|, \quad u_2 \in Y \qquad \Longrightarrow \qquad u_1 \in Y, \quad \|u_1\|_Y \le \|u_2\|_Y. \tag{53}$$

The general theory of ideal spaces (in the case of normed spaces) was considered in [3] and a special version of Banach function spaces, including Orlicz spaces, was developed in [11]. Let $P\colon M^+ \to L^+$ be a so-called monotone operator, i.e.,

$$f, h \in M^+, \quad f \le h \quad \mu\text{-almost everywhere} \qquad \Longrightarrow \qquad Pf \le Ph \quad \eta\text{-almost everywhere}. \tag{54}$$

For the cones Ω (39) and $\widetilde{\Omega}$ (42), we define the norms of the restrictions of the operator P as follows:

$$\|P\|_{\Omega\to Y} = \sup\{\|Pf\|_Y : f \in \Omega,\ \|f\|_{\varphi,v} \le 1\}, \tag{55}$$

$$\|P\|_{\widetilde{\Omega}\to Y} = \sup\{\|Pf\|_Y : f \in \widetilde{\Omega},\ \|f\|_{\varphi,v} \le 1\}. \tag{56}$$

Lemma 2. *Let $b > 1$, and let $\varphi \in A$. Suppose that the weight function satisfies conditions* (8) *and* (9) *and the discretization procedure* (40)–(46) *is applied to a function $f \in \Omega$. Then the following estimates hold*:

$$\|P\|_{\widetilde{\Omega}\to Y} \le \|P\|_{\Omega\to Y} \le d(b)\|P\|_{\widetilde{\Omega}\to Y}, \tag{57}$$

where $d(b)$ is defined in (1) *for $c = b > 1$.*

Proof. The left inequality in (57) is obvious in view of the embedding $\widetilde{\Omega} \subset \Omega$. On the other hand, if the function f_1 is of the form (43), then, for each function $f \in \Omega$, we have

$$f \le f_1 \quad \Longrightarrow \quad Pf \le Pf_1, \qquad \text{and} \qquad \|f_1\|_{\varphi,v} \le d(b)\|f\|_{\varphi,v}$$

(see Conclusion 6). Further,

$$f \in \Omega \quad \Longrightarrow \quad f_1 = \sum_m f(\mu_m)\chi_{\Delta_m} \in \widetilde{\Omega}.$$

Therefore, for each $f \in \Omega$,

$$\begin{aligned} \|Pf\|_Y \le \|Pf_1\|_Y \le \|P\|_{\widetilde{\Omega}\to Y}\|f_1\|_{\varphi,v} \le d(b)\|P\|_{\widetilde{\Omega}\to Y}\|f\|_{\varphi,v}, \\ \|P\|_{\Omega\to Y} = \sup\{\|Pf\|_Y : f \in \Omega,\ \|f\|_{\varphi,v} \le 1\} \le d(b)\|P\|_{\widetilde{\Omega}\to Y}. \end{aligned} \tag{58}$$

□

Now let us consider the norm of the restriction of the operator to the cone S (41):

$$\|P\|_{S\to Y} = \sup\{\|Pf\|_Y : f \in S,\ \|f\|_{\varphi,v} \le 1\}. \tag{59}$$

Theorem 1. *Suppose the assumptions of Lemma 2 hold. In the given notation, The following two-sided estimate is valid*:

$$c(b)^{-1}\|P\|_{S\to Y} \le \|P\|_{\Omega\to Y} \le d(b)\|P\|_{S\to Y}, \tag{60}$$

where $d(b)$ is defined in (1) for $c = b > 1$, and

$$c(b) = d(c_0(b)), \qquad c_0(b) = [b(b-1)^{-1}] > 1. \tag{61}$$

Proof. Inequality (60) follows from (57) and from the similar inequality

$$\|P\|_{\widetilde{\Omega}\to Y} \le \|P\|_{S\to Y} \le c(b)\|P\|_{\widetilde{\Omega}\to Y}. \tag{62}$$

The left inequality in (62) is obvious in view of the inclusion $\widetilde{\Omega} \subset S$. Let us prove the right inequality. We introduce the sup-operator A by the formula $A\gamma = \alpha$, where $\gamma = \{\gamma_m\}_{m\in\mathbb{Z}}$, $\alpha = \{\alpha_m\}_{m\in\mathbb{Z}}$, and

$$\alpha_m = \sup_{k\ge m} |\gamma_k|, \qquad m \in \mathbb{Z}. \tag{63}$$

We shall prove that the operator $A\colon l_{\varphi,\beta} \to l_{\varphi,\beta}$ is bounded. First, note that $\varphi \in C[0,\infty)$ increases, so that

$$|\gamma_k| \le \alpha_m \quad \text{for all} \quad k \ge m \qquad \Longrightarrow \qquad \sup_{k\ge m} \varphi(\lambda^{-1}|\gamma_k|) \le \varphi(\lambda^{-1}\alpha_m). \tag{64}$$

The reverse inequality is also valid. Indeed, for any $m \in \mathbb{Z}$ there are two possibilities:

(1) there exists a $k(m) \ge m$ such that $|\gamma_{k(m)}| = \alpha_m$;

(2) there exists a $\{k_j\}_{j\in N}$ such that $m \le k_j \uparrow +\infty$, $|\gamma_{k_j}| \uparrow \alpha_m$.

In case (1),

$$\varphi(\lambda^{-1}\alpha_m) = \varphi(\lambda^{-1}|\gamma_{k(m)}|) \le \sup_{k\ge m} \varphi(\lambda^{-1}|\gamma_k|). \tag{65}$$

In the case (2),

$$\varphi(\lambda^{-1}\alpha_m) = \lim_{j\to+\infty} \varphi(\lambda^{-1}|\gamma_{k_j}|) \le \sup_{k\ge m} \varphi(\lambda^{-1}|\gamma_k|).$$

Therefore,

$$\varphi(\lambda^{-1}\alpha_m) = \sup_{k\ge m} \varphi(\lambda^{-1}|\gamma_k|) \le \sum_{k\ge m} \varphi(\lambda^{-1}|\gamma_k|). \tag{66}$$

Then

$$j_\lambda(\alpha) = \sum_{m\in\mathbb{Z}} \varphi(\lambda^{-1}\alpha_m)\beta_m \le \sum_{m\in\mathbb{Z}} \beta_m \sum_{k\ge m} \varphi(\lambda^{-1}|\gamma_k|) = \sum_{k\in\mathbb{Z}} \varphi(\lambda^{-1}|\gamma_k|) \sum_{m\le k} \beta_m.$$

By (46), we have $\beta_m = b^{m+1} - b^m$, so that

$$\sum_{m\le k} \beta_m = b^{k+1} = \beta_k c_0(b) \tag{67}$$

in view of relation (61). Therefore,

$$j_\lambda(\alpha) \le c_0(b) \sum_{k\in\mathbb{Z}} \varphi(\lambda^{-1}|\gamma_k|)\beta_k = c_0(b) j_\lambda(\gamma). \tag{68}$$

This inequality, together with (29) and (30), gives

$$\|\alpha\|_{l_{\varphi,\beta}} \le c(b)\|\gamma\|_{l_{\varphi,\beta}}, \qquad c(b) = d(c_0(b)). \tag{69}$$

Now, for each $f \in S$, denote $\gamma = \{\gamma_m\}$, $\gamma_m = f(\mu_m) \ge 0$, $m \in \mathbb{Z}$. Then

$$f = f_{(\gamma)} := \sum_m \gamma_m \chi_{\Delta_m}.$$

Further, we introduce $\alpha_m = \sup_{k\ge m} \gamma_k$, $m \in \mathbb{Z}$ and, for $\alpha = \{\alpha_m\}$, consider the function

$$f_{(\alpha)} = \sum_m \alpha_m \chi_{\Delta_m}.$$

Then $f_{(\alpha)} \in \widetilde{\Omega}$ (see (42)) and

$$f_{(\alpha)} \ge f_{(\gamma)}, \qquad \|f_{(\alpha)}\|_{\varphi,v} = \|\alpha\|_{l_{\varphi,\beta}} \le c(b)\|\gamma\|_{l_{\varphi,\beta}} = c(b)\|f_{(\gamma)}\|_{\varphi,v} \tag{70}$$

(see (69)). It follows from (70) that, for $f = f_{(\gamma)} \in S$, there exists an $f_{(\alpha)} \in \widetilde{\Omega}$ such that

$$Pf \le Pf_{(\alpha)}, \qquad \|f_{(\alpha)}\|_{\varphi,v} \le c(b)\|f\|_{\varphi,v}.$$

Note that $f_{(\alpha)} \in \widetilde{\Omega}$; hence, for any function $f \in S$, we have

$$\|Pf\|_Y \le \|Pf_{(\alpha)}\|_Y \le \|P\|_{\widetilde{\Omega}\to Y}\|f_{(\alpha)}\|_{\varphi,v} \le c(b)\|P\|_{\widetilde{\Omega}\to Y}\|f\|_{\varphi,v}.$$

This yields the second inequality in (62). $\square$

Remark 7. Theorem 1 illustrates the main purpose the discretization (40)–(46). In this theorem, the estimates of the restriction of the monotone operator to the cone of decreasing functions Ω are reduced to the estimates of this operator on the set of all nonnegative step-functions. In a number of cases, the resulting reduction allows one to use well-known results for step-functions (or their discrete analogs) to obtain the required estimates on the cone Ω. This approach is realized in Sec. 4 using the description of associated norms as an example.

4. ASSOCIATED NORMS FOR THE CONE OF NONNEGATIVE DECREASING FUNCTIONS IN A WEIGHTED ORLICZ SPACE

In this section, we shall preserve the notation used in Secs. 1–3 and apply results from Sec. 3 to the important particular case in which the ideal space Y coincides with the weighted Lebesgue space $L_1(R_+; g)$, $g \in M^+$, and the monotone operator P is the identity operator. In this case,

$$\begin{aligned}\|P\|_{\Omega\to Y} &= \sup\left\{\int_0^\infty fg\,dt : f \in \Omega;\ \|f\|_{\varphi,v} \le 1\right\}\\ &= \sup\left\{\int_0^\infty fg\,dt : f \in \Omega;\ J_1(f) \le 1\right\} = \|g\|'\end{aligned} \tag{71}$$

(see (55); also recall that the equivalence $\|f\|_{\varphi,v} \le 1$, $\Leftrightarrow$, $J_1(f) \le 1$ holds; see (21)). This means that, in this case, the norm $\|P\|_{\Omega\to Y}$ coincides with the associated norm for the cone Ω (39) equipped with the functional

$$J_1(f) = \int_0^\infty \varphi(f)v\,dt.$$

By Theorem 1, we have

$$\|P\|_{\Omega\to Y} \cong \|P\|_{S\to Y}, \tag{72}$$

where, in our case,

$$\|P\|_{S\to Y} = \sup\left\{\sum_{m\in\mathbb{Z}} \alpha_m g_m : \alpha_m \geq 0,\ \sum_{m\in\mathbb{Z}} \varphi(\alpha_m)\beta_m \leq 1\right\} \tag{73}$$

for

$$g_m = \int_{\Delta_m} g\,dt \geq 0, \qquad \beta_m = \int_{\Delta_m} v\,dt = b^m(b-1), \quad m \in \mathbb{Z}. \tag{74}$$

Note that the norm (73) coincides with the discrete version of the Orlicz norm (see [2]):

$$\|\{g_m\}\|_{l'_{\varphi,\beta}} = \sup\left\{\sum_{m\in\mathbb{Z}} \alpha_m |g_m| : \alpha_m \geq 0,\ \sum_{m\in\mathbb{Z}} \varphi(\alpha_m)\beta_m \leq 1\right\}. \tag{75}$$

Our present goal is to describe the norm (75) in explicit form in terms of an additional function ψ. We shall restrict ourselves to the case of N-functions; see [2, Chap. 8]. Thus, let φ be an *N-function*, i.e., $\varphi \in C[0,\infty)$,

$$\varphi(s) = \int_0^s p(\sigma)\,d\sigma, \qquad s \in \mathbb{R}_+, \tag{76}$$

where p is an increasing right-continuous function, with $p(0) = 0$, $p(+\infty) = \infty$. Let ψ be an additional function, i.e.,

$$\psi(t) = \int_0^t q(\tau)\,d\tau, \qquad t \in \mathbb{R}_+, \quad q(\tau) = \sup\{\sigma : p(\sigma) \leq \tau\}. \tag{77}$$

The function q possesses the same properties as p, so that $\psi \in \mathbb{N}$. It is well known that

$$\psi(t) = \sup_{s\geq 0}[st - \varphi(s)], \qquad st \leq \varphi(s) + \psi(t), \quad s,t \in \mathbb{R}_+ \tag{78}$$

(the equality holds if and only if $p(s) = t$ or $q(t) = s$).

The following well-known result of the theory of discrete weighted Orlicz spaces holds for any positive weighted sequence. It plays a key role in the equivalent description of the Orlicz norm (75).

Theorem 2. *Let φ and ψ be additional N-functions, and let $\beta = \{\beta_m\}$, $\beta_m \in \mathbb{R}_+$, $m \in \mathbb{Z}$. Then the Orlicz norm* (75) *is equivalent to the norm*

$$\|\{g_m\}\|_{\widetilde{l}_{\psi,\beta}} := \|\{\beta_m^{-1} g_m\}\|_{l_{\psi,\beta}}. \tag{79}$$

Namely,

$$\|\{g_m\}\|_{\widetilde{l}_{\psi,\beta}} \leq \|\{g_m\}\|_{l'_{\varphi,\beta}} \leq 2\|\{g_m\}\|_{\widetilde{l}_{\psi,\beta}}. \tag{80}$$

Remark 8. Let us summarize our studies.

Let φ and ψ be additional N-functions, let conditions (8) and (9) hold, and let the discretization procedure (40)–(46) be applied. Then the norm (71) satisfies the following equivalence:

$$\|g\|' \cong \|\{\rho_m\}\|_{l_{\psi,\beta}}, \qquad \beta = \{\beta_m\}, \qquad \rho_m = \beta_m^{-1} \int_{\Delta_m} |g|\,dt. \tag{81}$$

Now our goal is to express this answer in integral form.

Theorem 3. *Let φ and ψ be additional N-functions, and let conditions* (8) *and* (9) *hold. For a fixed number $0 < a < 1$, the following two-sided estimate of the associated norm* (71) *holds*:

$$\|g\|' \cong \|\rho_a(g)\|_{\psi,v} = \inf\left\{\lambda > 0 : \int_0^\infty \psi(\lambda^{-1}\rho_a(g;t))v(t)\,dt \leq 1\right\}, \tag{82}$$

$$\rho_a(g;t) := V(t)^{-1}\int_{\delta_a(t)}^{t} |g(\tau)|\,d\tau, \quad \delta_a(t) := V^{-1}(aV(t)), \qquad t \in \mathbb{R}_+. \tag{83}$$

For different values of $a \in (0,1)$, the norms (82) are equivalent.

Here and elsewhere, we use the notation

$$A \cong B \qquad \Longleftrightarrow \qquad \text{there exists a} \quad c = c(a) \in [1,\infty) \quad \text{such that} \quad c^{-1} \le A/B \le c. \tag{84}$$

Remark 9. In addition, suppose that the function φ from Theorem 3 satisfies the Δ_2-condition, i.e.,

$$\text{there exists} \quad C \in (1,\infty) : \varphi(2t) \le C\varphi(t) \qquad \text{for all} \quad t \in \mathbb{R}_+. \tag{85}$$

Then

$$\|g\|' \cong \left\| V(t)^{-1}\int_0^t |g(\tau)|\,d\tau \right\|_{\psi,v}. \tag{86}$$

Proof of Theorem 3. We use the description (81) with $b = a^{-1/2} > 1$. Then $a = b^{-2}$ and

$$\rho'_m \le \rho_a(g;t) = V(t)^{-1}\int_{V^{-1}(aV(t))}^{t} |g|\,d\tau \le \rho''_m, \qquad t \in \Delta_m, \tag{87}$$

where

$$\rho'_m = b^{-(m+1)}\int_{\mu_{m-1}}^{\mu_m} |g|\,d\tau, \qquad \rho''_m = b^{-m}\int_{\mu_{m-2}}^{\mu_{m+1}} |g|\,d\tau. \tag{88}$$

Therefore,

$$F_0(t) \le \rho_a(g;t) \le F_1(t), \qquad t \in \mathbb{R}_+, \tag{89}$$

where F_0, F_1 are the step-functions

$$F_0(t) = \sum_m \rho'_m \chi_{\Delta_m}(t), \qquad F_1(t) = \sum_m \rho''_m \chi_{\Delta_m}(t),$$
$$\|F_0\|_{\psi,v} = \|\{\rho'_m\}\|_{l_{\psi,\beta}}, \qquad \|F_1\|_{\psi,v} = \|\{\rho''_m\}\|_{l_{\psi,\beta}},$$

so that

$$\|\{\rho'_m\}\|_{l_{\psi,\beta}} \le \|\rho_a(g)\|_{\psi,v} \le \|\{\rho''_m\}\|_{l_{\psi,\beta}}. \tag{90}$$

Thus, the required result (82) will follow from the equivalence

$$\|\{\rho'_m\}\|_{l_{\psi,\beta}} \cong \|\{\rho''_m\}\|_{l_{\psi,\beta}} \cong \|\{\rho_m\}\|_{l_{\psi,\beta}}. \tag{91}$$

It remains to prove (91). Relations (81) and (88) show that

$$\rho'_m = b^{-2}(b-1)\rho_{m-1}, \tag{92}$$
$$\rho''_m = \rho'_{m-1} + b\rho'_m + (b-1)\rho_m. \tag{93}$$

Therefore,

$$\|\{\rho'_m\}\|_{l_{\psi,\beta}} = b^{-2}(b-1)\|\{\rho_{m-1}\}\|_{l_{\psi,\beta}} \le b^{-1}(b-1)\|\{\rho_m\}\|_{l_{\psi,\beta}}, \tag{94}$$
$$\|\{\rho_m\}\|_{l_{\psi,\beta}} = b^2(b-1)^{-1}\|\{\rho'_{m+1}\}\|_{l_{\psi,\beta}} \le b^2(b-1)^{-1}\|\{\rho'_m\}\|_{l_{\psi,\beta}}. \tag{95}$$

In the last inequality, we took into account the fact that shift operators in the space $l_{\psi,\beta}$ with N-function ψ are bounded and $\beta = \{\beta_m\}$ from (46) (see Remark 5 and Lemma 1). Thus,

$$\|\{\rho_{m-1}\}\|_{l_{\psi,\beta}} \le b\|\{\rho_m\}\|_{l_{\psi,beta}}, \qquad \|\{\rho'_{m+1}\}\|_{l_{\psi,\beta}} \le \|\{\rho'_m\}\|_{l_{\psi,\beta}}.$$

In view of (93), we have

$$(b-1)\|\{\rho_m\}\|_{l_{\psi,\beta}} \le \|\{\rho''_m\}\|_{l_{\psi,\beta}}, \tag{96}$$

$$\|\{\rho''_m\}\|_{l_{\psi,\beta}} \le \|\{\rho'_{m-1}\}\|_{l_{\psi,\beta}} + b\|\{\rho'_m\}\|_{l_{\psi,\beta}} + (b-1)\|\{\rho_m\}\|_{l_{\psi,\beta}}. \tag{97}$$

Just as (94), the following estimate holds:

$$\|\{\rho'_{m-1}\}\|_{l_{\psi,\beta}} \le b\|\{\rho'_m\}\|_{l_{\psi,\beta}}.$$

Substituting this estimate into (97) and noting inequality (94), we obtain

$$\|\{\rho''_m\}\|_{l_{\psi,\beta}} \le 3(b-1)\|\{\rho_m\}\|_{l_{\psi,\beta}}.$$

Therefore,

$$(b-1)\|\{\rho_m\}\|_{l_{\psi,\beta}} \le \|\{\rho''_m\}\|_{l_{\psi,\beta}} \le 3(b-1)\|\{\rho_m\}\|_{l_{\psi,\beta}}. \tag{98}$$

Estimates (94), (95), and (98) imply that the required equivalence (91). □

5. APPLICATIONS TO WEIGHTED ORLICZ–LORENTZ CLASSES

Let us recall the notion of decreasing rearrangement of a function. Let $M_0 = M_0(\mathbb{R}_+)$ be the subspace of all functions $f\colon \mathbb{R}_+ \to \mathbb{R}$ measurable with respect to Lebesgue measure μ, finite almost everywhere and such that, for $f \in M_0$, the distribution function λ_f is not identical to infinity, where

$$\lambda_f(y) = \mu\{x \in \mathbb{R}_+ : |f(x)| > y\}, \qquad y \in \mathbb{R}_+. \tag{99}$$

Then $0 \le \lambda_f \downarrow$, $\lambda_f(y) \to 0$, $y \to +\infty$.

Let f^* be the decreasing rearrangement of the function f, i.e.,

$$f^*(t) = \inf\{y \in \mathbb{R}_+ : \lambda_f(y) \le t\}, \qquad t \in \mathbb{R}_+. \tag{100}$$

We consider the Orlicz–Lorentz version $\Lambda_{\varphi,v}$ for the Orlicz space $L_{\varphi,v}$. For $f \in M_0$, we define

$$J_\lambda(f^*) = \int_0^\infty \varphi(\lambda^{-1} f^*(t)) v(t)\, dt, \qquad \lambda > 0, \tag{101}$$

where $v \in M^+$, the integration is over Lebesgue measure, and the weight function satisfies condition (8). The weighted Orlicz–Lorentz class $\Lambda_{\varphi,v}$ consists of functions $f \in M_0(\mathbb{R}_+)$ such that $f^* \in L_{\varphi,v}$. This class is equipped with the functional

$$\|f^*\|_{\varphi,v} = \inf\{\lambda > 0 : J_\lambda(f^*) \le 1\}. \tag{102}$$

For the space $\Lambda_{\varphi,v}$ to be linear, we additionally assume that the weight function V (8) satisfies the Δ_2-condition, i.e.,

$$\text{there exists a} \quad C \in \mathbb{R}_+ : V(2t) \le CV(t) \qquad \text{for all} \quad t \in \mathbb{R}_+. \tag{103}$$

It is known that this assumption is necessary and sufficient for the validity of the triangle inequality in the Lorentz space (see, for example, [14]). However, *we do not need estimate* (103) *in our study*. In any case, we can consider the class $\Lambda_{\varphi,v}$ as a cone in M_0 consisting of functions having finite values of the functional (102). Here we present analogs of results from Sec. 3 concerning estimates of the norms of monotone operators over Orlicz-Lorentz classes. Recall some descriptions. Let (N, η) be a space with nonnegative σ-finite measure η. By $L = L(N, \eta)$ we shall denote the space of all η-measurable functions $u\colon N \to \mathbb{R}$, and let $L^+ = \{u \in L : u \ge 0\}$. Let $Y_i = Y_i(N, \eta) \subset L$, $i = 1, 2$, be ideal spaces, and let $P\colon M_0^+(\mathbb{R}_+) \to L^+$ be the monotone operator related to these spaces by the following condition: for $h \in \Omega$,

$$\|Ph\|_{Y_2} = \sup\{\|Pf\|_{Y_1} : f \in M_0^+(R_+),\ f^* = h\}. \tag{104}$$

Let us illustrate these conditions by two examples.

Example 5. Suppose that P is the identity operator on $M_0^+(\mathbb{R}_+)$ and

$$Y_1 = L_1(\mathbb{R}_+; g), \quad g \in M_0^+(\mathbb{R}_+), \qquad Y_2 = L_1(\mathbb{R}_+; g^*).$$

Then, relation (104) reflects the well-known extremal property of decreasing rearrangements (see [11, Secs. 2.3–2.8])

$$\sup\left\{\int_0^\infty |fg|\, dt : f \in M_0,\ f^* = h\right\} = \int_0^\infty hg^*\, dt.$$

Example 6. Let Y be an ideal space, and le $P\colon M_0^+(\mathbb{R}_+) \to L^+$ be the monotone operator satisfying the condition

$$\|Pf\|_Y \le \|Pf^*\|_Y, \qquad f \in M_0^+(\mathbb{R}_+). \tag{105}$$

Then relation (104) holds with $Y_1 = Y_2 = Y$.

Indeed, $f \in M_0^+(\mathbb{R}_+)$, $\Rightarrow$, $h := f^* \in M_0^+(\mathbb{R}_+)$, $h^* = h$, and

$$\|Ph\|_Y \le \sup\{\|Pf\|_Y : f \in M_0^+(\mathbb{R}_+),\ f^* = h\}.$$

On the other hand, for any function $f \in M_0^+(\mathbb{R}_+)$: $f^* = h$, by (105), we have

$$\|Pf\|_Y \le \|Pf^*\|_Y = \|Ph\|_Y \quad \Longrightarrow \quad \sup\{\|Pf\|_Y : f \in M_0^+(\mathbb{R}_+),\ f^* = h\} \le \|Ph\|_Y.$$

Example 7. In particular, Example 6 includes operators of the form

$$(Pf)(x) = \int_0^\infty k(x,\tau) f(\tau)\, d\tau, \qquad x \in \mathbb{N}, \tag{106}$$

where k is a nonnegative measurable function on $\mathbb{N}\times\mathbb{R}_+$, and $k(x,\tau)$ is a decreasing and right-continuous function of $\tau \in \mathbb{R}_+$. Then, for almost all $x \in \mathbb{N}$, by the well-known Hardy lemma, we have

$$|(Pf)(x)| \le \int_0^\infty k(x,\tau)|f(\tau)|\, d\tau \le \int_0^\infty k(x,\tau) f^*(\tau)\, d\tau = |(Pf^*)(x)|.$$

Therefore, for such operators, inequality (105) holds for any ideal space Y.

Proposition 6. *Let relation* (104) *in the notation of this section hold. Then the norms*

$$\|P\|_{\Lambda_{\varphi,v}\to Y_1} = \sup\{\|Pf\|_{Y_1} : f \in M_0^+(\mathbb{R}_+),\ \|f^*\|_{\varphi,v} \le 1\}, \tag{107}$$
$$\|P\|_{\Omega\to Y_2} = \sup\{\|Ph\|_{Y_2} : h \in \Omega,\ \|h\|_{\varphi,v} \le 1\}. \tag{108}$$

coincide:

$$\|P\|_{\Lambda_{\varphi,v}\to Y_1} = \|P\|_{\Omega\to Y_2}. \tag{109}$$

Proof. Using the equivalence

$$f \in M_0, \qquad \|f^*\|_{\varphi,v} \le 1 \quad \Longleftrightarrow \quad h = f^* \in \Omega : \|h\|_{\varphi,v} \le 1,$$

we obtain

$$\|P\|_{\Lambda_{\varphi,v}\to Y_1} = \sup\big[\sup\{\|Pf\|_{Y_1} : f \in M_0^+(\mathbb{R}_+),\ f^* = h\} : h \in \Omega,\ \|h\|_{\varphi,v} \le 1\big].$$

By (104), the right-hand side of this equality coincides with

$$\sup[\|Ph\|_{Y_2} : h \in \Omega,\ \|h\|_{\varphi,v} \le 1] = \|P\|_{\Omega\to Y_2}.$$

□

Remark 10. This proposition allows us to reduce the estimates of the norm $\|P\|_{\Lambda_{\varphi,v}\to Y_1}$ (107) to the estimates given in Secs. 3 and 4. In particular, using Example 5, we can reduce the calculation of the associated norm for a function $g \in M$ on the Orlicz–Lorentz class to the associated norm on the cone Ω for its decreasing rearrangement g^*:

$$\|g\|_*' := \sup\left\{\int_0^\infty |fg|\, dt : f \in M_0;\ \|f^*\|_{\varphi,v} \le 1\right\} = \|g^*\|'.$$

Then Theorem 3 and Remark 9 lead to the following result.

Theorem 4. *Suppose that the assumptions of Theorem* 3 *hold. Then*

$$\|g\|_*^{/} \cong \|\rho_a(g^*)\|_{\psi,v} = \inf\left\{\lambda > 0 : \int_0^\infty \psi(\lambda^{-1}\rho_a(g^*;t))v(t)\, dt \le 1\right\}, \tag{110}$$

where ρ_a *was defined in* (83). *For different values of* $a \in (0,1)$, *the norms* (110) *are equivalent.*

Remark 11. In addition, suppose that, the function φ in Theorem 4 satisfies the Δ_2-condition. Then

$$\|g\|_*' \cong \left\| V(t)^{-1} \int_0^t g^*(\tau)\, d\tau \right\|_{\psi,v}. \tag{111}$$

Remark 12. Relations (110) and (111) modify results from [12], in which some previous results from [13] were developed. In particular, formula (111) was established in [13] under the assumption that both functions φ and ψ *satisfy the* Δ_2*-condition.* For duality for Orlicz and Orlicz–Lorentz spaces, see also [2], [4], [15], [16].

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BIBLIOGRAPHY

1. M. A. Krasnosel'skii and Ya. B. Rutitskii, *Convex Functions and Orlicz Spaces*, in *Contemporary Problems of Mathematics* (Fizmatgiz, Moscow, 1958) [in Russian].
2. L. Maligranda, *Orlicz Spaces and Interpolation*, in *Seminários de Matematica* (Univ. Estadual de Campinas, Departamento de Matemática, Campinas, 1989), Vol. 5.
3. S. G. Krein, Yu. I. Petunin, and E. M. Semenov, *Interpolation of Linear Operators* (Nauka, Moscow, 1978) [in Russian].
4. H. Hudzik, A. Kaminska, and M. Mastylo, "On the dual of Orlicz–Lorentz space," Proc. Amer. Math. Soc. **130** (6), 1645–1654 (2002).
5. V. I. Ovchinnikov, "Interpolation in quasi-Banach Orlicz spaces," Funktsional. Anal. Prilozhen. **16** (3), 78–79 (1982) [Functional Anal. Appl. **16**, 223–224 (1983)].
6. P. P. Zabreiko, "An interpolation theorem for linear operators," Mat. Zametki **2** (6), 593–598 (1967).
7. L. V. Kantorovich and G. P. Akilov, *Functional Analysis* (Nauka, Moscow, 1984) [in Russian].
8. G. Ya. Lozanovskii, "Certain Banach lattices," Sibirsk. Mat. Zh. **10**, 584–599 (1969) [Sib. Math. J. **10**, 419–431 (1969)].
9. G. Ya. Lozanovskii, "Certain Banach lattices. II," Sibirsk. Mat. Zh. **12**, 562–567 (1971) [Sib. Math. J. **12**, 397–401 (1971)].
10. V. I. Ovchinnikov, "The method of orbits in interpolation theory," Math. Rep. **1** (2), 349–515 (1984).
11. C. Bennett and R. Sharpley, *Interpolation Operators*, in *Pure Appl. Math.* (Acad. Press, Boston, MA, 1988), Vol. 129.

12. M. L. Goldman and R. Kerman, "On the principal of duality in Orlicz–Lorentz spaces," in *Function Spaces. Differential Operators. Problems of Mathematical Education*, Proc. Intern. Conf. Dedicated to the 75th Birthday of Prof. L. D. Kudrjavtsev (Moscow, 1998), Vol. 1, pp. 179–183.
13. H. Heinig and A. Kufner, "Hardy operators on monotone functions and sequences in Orlicz Spaces," J. London Math. Soc. (2) **53** (2), 256–270 (1996).
14. A. Kaminska and L. Maligranda, "Order convexity and concavity in Lorentz spaces $\Lambda_{p,w}$, $0 < p < \infty$," Studia Math. **160** (3), 267–286 (2004).
15. A. Kaminska and M. Mastylo, "Abstract duality Sawyer formula and its applications," Monatsh. Math. **151** (3), 223–245 (2007).
16. A. Kaminska and Y. Raynaud, "New formula for decreasing rearrangements and the class of Orlicz–Lorentz spaces," Rev. Mat. Complut. **27** (2), 587–621 (2014).
17. E. Sawyer, "Boundedness of classical operators on classical Lorentz spaces," Studia Math. **96** (2), 145–158 (1990).

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