MATHEMATICAL NOTES, VOL. 0, NO. 0, 1966, P. 1.
Translated from Matematicheskie Zametki, vol. 0, *no.* 0, 1966, *p.* 0.

Positive Radially Symmetric Solution of the Dirichlet Problem for a Nonlinear Elliptic System with p-Laplacian

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Received December 2, 2015

Abstract—Sufficient conditions for the existence and uniqueness of a positive radially symmetric solution of the Dirichlet problem for a nonlinear elliptic second-order system with p-Laplacian are obtained. In addition it also proved that these conditions guarantee the nonexistence of a global positive radially symmetric solution.

KEY WORDS: *nonlinear differential equation*, *Dirichlet problem*, *p-Laplacian*, *positive solution.*

1. INTRODUCTION {ssec1}

In the ball $S = \{x \in \mathbb{R}^n : |x| < 1\}$, consider the Dirichlet problem

$$\Delta_p u_1 + |x|^{m_2}|u_2|^{q_2} = 0, \qquad x \in S, \tag{1}$$ {eq1.1}

$$\Delta_p u_2 + |x|^{m_1}|u_1|^{q_1} = 0, \qquad x \in S, \tag{2}$$ {eq1.2}

$$u_i|_{\Gamma=0}, \qquad i = 1, 2, \tag{3}$$ {eq1.3}

where Γ is the boundary of the ball S, $m_i \geq 0$, $q_i > 1$, $i = 1, 2$, and $p > 1$ are constants, and

$$\Delta_p v = \operatorname{div}(|\nabla v|^{p-2}\nabla v).$$

Obviously, $u = (u_1, u_2) \equiv 0$ is the trivial solution of this problem. By a positive solution of problem (1)–(3) we mean a function $u = (u_1, u_2)$ such that it satisfies $u_i \in C^2(\overline{S})$, $i = 1, 2$, Eqs. (1), (2), and boundary conditions (3), and is positive in S.

Positive solutions of the Dirichlet problem for nonlinear elliptic equations are considered in many papers (see [1]–[12] and the literature cited therein). Many of these deal with questions of the existence and nonexistence of positive solutions, as well as with a priori estimates, the global and local behavior of positive solutions, their uniqueness, etc. Note that there are only a few papers dealing with uniqueness of positive solutions.

The goal in the present paper is to obtain sufficient conditions for the existence and uniqueness of a positive radially symmetric solution of problem (1)–(3) and for the nonexistence of a global positive solution.

Earlier the questions of the existence and uniqueness of a positive radially symmetric solution of problem (1)–(3) for $p = 2$ were studied in the author's paper [12]. In it, it was shown that, in this case, the condition

$$1 < q_i \leq \frac{n + m_i}{n - 2} \tag{4}$$ {eq1.4}

guarantees the existence of a unique positive radially symmetric solution of problem (1)–(3). In the author's paper [11], it was shown that, in the case of one equation with p-Laplacian,

$$\Delta_p u + |x|^m |u|^q = 0, \quad x \in S, \qquad m \geq 0, \quad 1 < p \leq 2, \tag{5}$$ {eq1.5}

the condition

$$1 < q \le \frac{(p-1)(n+m)}{n-p} \qquad (6) \quad \{eq1.6\}$$

guarantees the existence of a unique positive radially symmetric solution.

In the present paper, these results are generalized to the case of system (1), (2). In addition, it is shown that the same conditions ensure the existence of a unique positive radially symmetric solution of problem (1)–(3) and the nonexistence of a global positive radially symmetric solution of system (1), (2). In other words, the existence of a unique positive radially symmetric solution of problem (1)–(3) and the nonexistence of a global nontrivial solution of system (1), (2) are ensured by the same conditions. A similar fact was noted by H. Zou, who showed in [6] that, for the Lane–Emden equation $\Delta u + u^q = 0$, the existence of a positive solution in a bounded smooth domain Ω is equivalent to the nonexistence of a global positive solution of this equation.

2. EXISTENCE AND UNIQUENESS OF A POSITIVE RADIALLY SYMMETRIC SOLUTION FOR $n \ge 2$

{ssec2}

If there exists a radially symmetric solution $u = u(r)$ of problem (1), (2), then this solution satisfies the following system of ordinary differential equations:

$$|u_1'|^{p-2}\left(u_1'' + \frac{n-1}{r}u_1'\right) + r^{m_2}|u_2|^{q_2} = 0, \qquad 0 < r < 1, \qquad (7) \quad \{eq2.1\}$$

$$|u_2'|^{p-2}\left(u_2'' + \frac{n-1}{r}u_2'\right) + r^{m_1}|u_1|^{q_1} = 0, \qquad 0 < r < 1, \qquad (8) \quad \{eq2.2\}$$

where $r = |x|$, and also the boundary conditions

$$u_i'(0) = 0, \quad u_i(1) = 0, \qquad i = 1, 2. \qquad (9) \quad \{eq2.3\}$$

Using the linear transformation of T. Na (see [13])

$$r = A^{\alpha}\overline{r}, \qquad u_i = A^{\beta_i}\overline{u}_i, \quad i = 1, 2, \qquad (10) \quad \{eq2.4\}$$

where α, β_1, and β_2 are constants and A is a positive transformation parameter, we can write system (7), (8) in the form

$$A^{(\beta_1-\alpha)(p-2)}|\overline{u}_1|^{p-2}\left((p-1)A^{\beta_1-2\alpha}\overline{u}_1'' + \frac{n-1}{\overline{r}}\overline{u}_1'\right) + +A^{\alpha m_2+\beta_2 q_2}\overline{r}^{m_2}|\overline{u}_2|^{q_2} = 0,$$

$$A^{(\beta_2-\alpha)(p-2)}|\overline{u}_2|^{p-2}\left((p-1)A^{\beta_2-2\alpha}\overline{u}_2'' + \frac{n-1}{\overline{r}}\overline{u}_2'\right) + +A^{\alpha m_1+\beta_1 q_1}\overline{r}^{m_1}|\overline{u}_1|^{q_1} = 0.$$

Let us choose α and β_i, $i = 1, 2$, so that these equations do not depend on the parameter A. To this end, it suffices to set

$$(\beta_1-\alpha)(p-2) + \beta_1 - 2\alpha = \alpha m_2 + \beta_2 q_2, \qquad (11) \quad \{eq2.5\}$$

$$(\beta_2-\alpha)(p-2) + \beta_2 - 2\alpha = \alpha m_1 + \beta_1 q_1. \qquad (12) \quad \{eq2.6\}$$

Then $\overline{u}_i$, $i = 1, 2$, will satisfy the equations

$$|\overline{u}_1|^{p-2}\left((p-1)\overline{u}_1'' + \frac{n-1}{\overline{r}}\overline{u}_1'\right) + \overline{r}^{m_2}|\overline{u}_2|^{q_2} = 0, \qquad (13) \quad \{eq2.7\}$$

$$|\overline{u}_2|^{p-2}\left((p-1)\overline{u}_2'' + \frac{n-1}{\overline{r}}\overline{u}_2'\right) + \overline{r}^{m_1}|\overline{u}_1|^{q_1} = 0. \qquad (14) \quad \{eq2.8\}$$

It has turned out that the system of equations (13), (14) is invariant with respect to the transformation (10). Obviously, the conditions $u_i'(0) = 0$, $i = 1, 2$, become the conditions

$$\overline{u}_i{}'(0) = 0, \qquad i = 1, 2. \tag{15}$$ {eq2.9}

The initial conditions $u_i(0)$, $i = 1, 2$, are nit enough for the Cauchy problem for system (7), (8). Denote them as follows:

$$u_1(0) = A, \qquad u_2(0) = B, \tag{16}$$ {eq2.10}

where A and B are positive numerical parameters. In the new coordinates $(\overline{r}, \overline{u})$ (see (10)), these conditions become

$$A^{\beta_1}\overline{u}_1(0) = A, \tag{17}$$ {eq2.11}

$$A^{\beta_2}\overline{u}_2(0) = B. \tag{18}$$ {eq2.12}

Equality (17) will be independent of the parameter A if we take

$$\beta_1 = 1. \tag{19}$$ {eq2.13}

Then conditions (17) and (18) imply the equalities

$$\overline{u}_1(0) = 1, \qquad \overline{u}_2(0) = \lambda,$$

where $\lambda = A^{-\beta_2}B$. Solving system (11), (12) and taking (19) into account, we obtain

$$\beta_2 = \frac{1}{p-1}\left[q_1 - \frac{(m_1+p)(q_1q_2-(p-1)^2)}{(m_2+p)(p-1)+(m_1+p)q_2}\right], \tag{20}$$ {eq2.14}

$$\alpha = -\frac{q_1q_2-(p-1)^2}{(m_2+p)(p-1)+(m_1+p)q_2}. \tag{21}$$ {eq2.15}

Thus, $\overline{u} = (\overline{u}_1, \overline{u}_2)$ is a solution of the Cauchy problem for Eqs. (13), (14) with initial conditions

$$\overline{u}_1(0) = 1, \qquad \overline{u}_2(0) = \lambda, \tag{22}$$ {eq2.16}

$$\overline{u}_1{}'(0) = \overline{u}_2{}'(0) = 0. \tag{23}$$ {eq2.17}

In what follows, we shall consider the case $1 < p \le 2$. The following statement holds.

{lem1}

Lemma 1. *If the following inequalities hold*:

$$1 < q_i \le \frac{(n+m_i)(p-1)}{n-p}, \qquad i = 1, 2, \tag{24}$$ {eq2.18}

where $1 < p \le 2$ for $n \ge 3$ and $1 < p < 2$ for $n = 2$, then, for any $\lambda > 0$, there exist unique positive numbers $\overline{r}_1, \overline{r}_2$ such that the Cauchy problem (13), (14), (22), (23) *has a unique solution $\overline{u}_i \in C^2[0, \overline{r}_i]$, $i = 1, 2$, with $\overline{u}_i(\overline{r}_i) = 0$, $\overline{u}_i(\overline{r}) > 0$, $i = 1, 2$, for $\overline{r} \in [0, \overline{r}_i)$.*

Proof. Let us rewrite Eqs. (13), (14) as

$$(p-1)\overline{u}_1{}'' + \frac{n-1}{\overline{r}}\overline{u}_1{}' = -\overline{r}^{m_2}|\overline{u}_2|^{q_2}|\overline{u}_1'|^{2-p}, \tag{25}$$ {eq2.19}

$$(p-1)\overline{u}_2{}'' + \frac{n-1}{\overline{r}}\overline{u}_2{}' = -\overline{r}^{m_1}|\overline{u}_1|^{q_1}|\overline{u}_2'|^{2-p}. \tag{26}$$ {eq2.20}

We introduce the new variable $t = \overline{r}^{(p-n)/(p-1)}$. If $n \ge 3$, $1 < p \le 2$ or $n = 2$, $1 < p < 2$, then the values of t, $0 < t < \infty$, correspond to those of $\overline{r}$, $0 < \overline{r} < \infty$, and $t \to \infty$ as $\overline{r} \to 0$ and $t \to 0$

as $\overline{r} \to \infty$. The case $p = 2$, $n = 2$ was studied in [12]. In this case, problem (1)–(3) has a unique positive radially symmetric solution for any $q_i > 1$, $i = 1, 2$. After this change, the Cauchy problem (13), (14), (22), (23) takes the form

$$v_1'' = -\frac{(p-1)^{p-1}}{(n-p)^p} t^{-(m_2(p-1)+p(n-1))/(n-p)} |v_2|^{q_2} |v_1'|^{2-p}, \tag{27}$$ {eq2.21}

$$v_2'' = -\frac{(p-1)^{p-1}}{(n-p)^p} t^{-(m_1(p-1)+p(n-1))/(n-p)} |v_1|^{q_1} |v_2'|^{2-p}, \tag{28}$$ {eq2.22}

$$\lim_{t\to\infty} v_1(t) = 1, \qquad \lim_{t\to\infty} v_2(t) = \lambda, \tag{29}$$ {eq2.23}

$$\lim_{t\to\infty} v_1'(t) = 0, \qquad \lim_{t\to\infty} v_2'(t) = 0, \tag{30}$$ {eq2.24}

where $v_i(t) = \overline{u}_i(t^{(p-1)/(p-n)})$, $i = 1, 2$.

Integrating Eqs. (27), (28) from t to ∞ and taking the initial conditions (30) into account, we obtain

$$v_1'(t) = \frac{(p-1)^{p-1}}{(n-p)^p} \int_t^\infty s^{-(m_2(p-1)+p(n-1))/(n-p)} |v_2(s)|^{q_2} |v_1'(s)|^{2-p} \, ds, \tag{31}$$ {eq2.25}

$$v_2'(t) = \frac{(p-1)^{p-1}}{(n-p)^p} \int_t^\infty s^{-(m_1(p-1)+p(n-1))/(n-p)} |v_1(s)|^{q_1} |v_2'(s)|^{2-p} \, ds. \tag{32}$$ {eq2.26}

This, together with Eqs. (27), (28), implies that $v_i(t)$, $i = 1, 2$, are convex (upward) and increasing functions. Therefore, there exist unique points t_1, t_2 such that $v_i(t_i) = 0$, $i = 1, 2$. Let us show that $t_i > 0$, $i = 1, 2$. Indeed, if we assume that this is not so then $v_i(0) \geq 0$. Let T be a positive number such that $v_i(T) > 0$. Since $v_i''(t) \leq 0$, $i = 1, 2$, in view of (27), (28), it follows that the functions $v_i'(t)$ decrease and $v_i'(t) > 0$, $i = 1, 2$, for $t \in [0, T]$. Therefore, for $t = 0$, from (31), (32), we obtain

$$v_1'(0) \geq \frac{(p-1)^{p-1}}{(n-p)^p} (v_1'(T))^{2-p} \int_0^T s^{-(m_2(p-1)+p(n-1))/(n-p)} |v_2(s)|^{q_2} \, ds, \tag{33}$$ {eq2.27}

$$v_2'(0) \geq \frac{(p-1)^{p-1}}{(n-p)^p} (v_2'(T))^{2-p} \int_0^T s^{-(m_1(p-1)+p(n-1))/(n-p)} |v_1(s)|^{q_1} \, ds. \tag{34}$$ {eq2.28}

Since the functions $v_i(t)$ are convex (upward) and increasing, it is easy to verify that, for $s \in [0, T]$, they satisfy the inequalities

$$v_i(s) \geq \frac{v_i(T)}{T} s, \qquad i = 1, 2.$$

Using these inequalities, from (33), (34), we obtain

$$v_1'(0) \geq \frac{(p-1)^{p-1}}{(n-p)^p} (v_1'(T))^{2-p} \left(\frac{v_2(t)}{T} \right)^{q_2} \int_0^T s^{q_2-(m_2(p-1)+p(n-1))/(n-p)} \, ds,$$

$$v_2'(0) \geq \frac{(p-1)^{p-1}}{(n-p)^p} (v_2'(T))^{2-p} \left(\frac{v_2(t)}{T} \right)^{q_2} \int_0^T s^{q_1-(m_1(p-1)+p(n-1))/(n-p)} \, ds.$$

In view of condition (24), the integrals on the right-hand sides of these inequalities are divergent. Therefore, the functions $v_i(t)$ cannot be a solution of problem (27)–(30) on $[0, \infty]$. We have obtained a contradiction. Therefore, there exist unique points $t_i > 0$, $i = 1, 2$ such that $v_i(t_i) = 0$ and $v_i(t) > 0$, $i = 1, 2$, for $t \in (t_i, \infty)$. Since $t = \overline{r}^{(p-n)/(p-1)}$, there exist unique points $\overline{r}_i > 0$, $i = 1, 2$ such that $\overline{u}_i(\overline{r}_i) = 0$ and $\overline{u}_i(\overline{r}) > 0$, $i = 1, 2$, for $\overline{r} \in [0, \overline{r}_i)$, where $\overline{r}_i = t_i^{(p-1)/(p-n)}$. Since

$$\overline{u}_i(\overline{r}) = -\frac{n-p}{p-1} v_i(t) \overline{r}^{-(n-1)/(p-1)} < 0,$$

in view of (22), we have

$$0 \le \overline{u}_1(\overline{r}) \le 1, \qquad 0 \le \overline{r} \le \overline{r}_1,$$
$$0 \le \overline{u}_2(\overline{r}) \le \lambda, \qquad 0 \le \overline{r} \le \overline{r}_2.$$

Let us now estimate the derivatives. In Eqs. (13), (14), we make the change $\overline{u}_2 = \lambda\overline{U}_2$. Then these equations become

$$(p-1)\overline{u}_1{}'' + \frac{n-1}{\overline{r}}\overline{u}_1' = -\overline{r}^{m_2}\lambda^{q_2}|\overline{U}_2|^{q_2}|\overline{u}_1'|^{2-p}, \tag{35} \quad \{eq2.29\}$$

$$(p-1)\overline{U}_2{}'' + \frac{n-1}{\overline{r}}\overline{U}_2' = -\frac{\overline{r}^{m_1}}{\lambda^{p-1}}|\overline{u}_1|^{q_1}|\overline{u}_2'|^{2-p}. \tag{36} \quad \{eq2.30\}$$

It follows from the condition $\overline{u}_2(0) = \lambda$ that

$$\overline{U}_2(0) = 1. \tag{37} \quad \{eq2.31\}$$

Multiplying Eqs. (35), (36) by $\overline{r}^{(n-1)/(p-1)}$, integrating the resulting equalities from 0 to $\overline{r}$, and taking the initial conditions (23) into account, we obtain

$$\overline{u}_1'(\overline{r}) = -\frac{\lambda^{q_2}}{(n-p)\overline{r}^{(n-1)/(p-1)}}\int_0^{\overline{r}} s^{m_2+(n-1)/(p-1)}|\overline{U}_2(s)|^{q_2}|\overline{u}_1'|^{2-p}\,ds, \tag{38} \quad \{eq2.32\}$$

$$\overline{U}_2'(\overline{r}) = -\frac{1}{\lambda^{p-1}(n-p)\overline{r}^{(n-1)/(p-1)}}\int_0^{\overline{r}} s^{m_1+(n-1)/(p-1)}|\overline{u}_1(s)|^{q_1}|\overline{U}_2'(s)|^{2-p}\,ds. \tag{39} \quad \{eq2.33\}$$

To be definite, suppose that $\overline{r}_1 \le \overline{r}_2$ for a fixed λ. Denote

$$\max_{0\le\overline{r}\le\overline{r}_2}|\overline{u}_1'(\overline{r})| = a.$$

Using (38) and taking into account the inequalities $0 \le \overline{U}_2(\overline{r}) \le 1$ for $0 \le \overline{r} \le \overline{r}_2$, we obtain

$$a^{p-1} \le \frac{\lambda^{q_2}}{n-p}\max_{0\le\overline{r}\le\overline{r}_2}\frac{\int_0^{\overline{r}} s^{m_2+(n-1)/(p-1)}\,ds}{\overline{r}^{(n-1)/(p-1)}} = \frac{\lambda^{q_2}}{n-p}\overline{r}_2^{m_2+1}. \tag{40} \quad \{eq2.34\}$$

Hence we have

$$a^{2-p} \le \frac{\lambda^{q_2(2-p)/(p-1)}}{(n-p)^{(2-p)/(p-1)}}\,\overline{r}_2^{(m2+1)(2-p)/(p-1)}. \tag{41} \quad \{eq2.35\}$$

Let us now integrate equality (38) from $\overline{r}_1$ to $\overline{r}_2$:

$$\overline{u}_1(\overline{r}_2) = -\frac{\lambda^{q_2}}{(n-p)}\int_{\overline{r}_1}^{\overline{r}_2}\tau^{-(n-1)/(p-1)}\int_0^{\tau} s^{m_2+(n-1)/(p-1)}|\overline{U}_2(s)|^{q_2}|\overline{u}_1'|^{2-p}\,ds\,d\tau.$$

Hence, in view of (41), we have

$$\overline{u}_1(\overline{r}_2) \le \frac{\lambda^{q_2/(p-1)}\overline{r}_2^{(m_2+1)(2-p)/(p-1)}}{(n-p)^{1/(p-1)}} \equiv c_0.$$

Therefore,

$$\overline{u}_1(\overline{r}) \le \max(1, c_0) \equiv M_0 \tag{42} \quad \{eq2.36\}$$

for $0 \le \overline{r} \le \overline{r}_2$. It follows from (40) that

$$|\overline{u}_1'(\overline{r})| \le \left(\frac{\lambda^{q_2}}{n-p}\overline{r}_2^{m_2+1}\right)^{1/(p-1)} \equiv M_1. \tag{43} \quad \{eq2.37\}$$

Denote
$$\max_{0\le\overline{r}\le\overline{r}_2}|\overline{U}_2'(\overline{r})| = b.$$
In view of (42), from (39), we obtain
$$b^{p-1}\le\frac{M_0^{q_1}}{\lambda^{p-1}(n-p)}\max_{0\le\overline{r}\le\overline{r}_2}\frac{\int_0^{\overline{r}}s^{m_1+(n-1)/(p-1)}\,ds}{\overline{r}^{(n-1)/(p-1)}}\le\frac{M_0^{q_1}}{\lambda^{p-1}(n-p)}\overline{r}_2^{m_1+1}.$$
Therefore,
$$|\overline{U}_2'(\overline{r})|\le\frac{M_0^{q_1/(p-1)}}{\lambda(n-p)^{1/(p-1)}}\overline{r}_2^{(m_1+1)/(p-1)}\equiv M_2. \qquad (44)$$ {eq2.38}

Thus, in view of (42)–(44), for $0\le\overline{r}\le\overline{r}_2$, the following estimates hold:
$$|\overline{u}_1(\overline{r})|\le M_0,\qquad|\overline{u}_1'(\overline{r})|\le M_1,\qquad|\overline{U}_2(\overline{r})|\le 1,\qquad\overline{U}_2'(\overline{r})|\le M_2. \qquad (45)$$ {eq2.39}

Let us now estimate the second derivatives. It follows from (38) that
$$\frac{\overline{u}_1'(\overline{r})}{\overline{r}}=-\frac{\lambda^{q_2}}{(n-p)\overline{r}^{(n+p-2)/(p-1)}}\int_0^{\overline{r}}s^{m_2+(n-1)/(p-1)}|\overline{U}_2(s)|^{q_2}|\overline{u}_1'|^{2-p}\,ds.$$
Hence, using estimates (45) for $0\le\overline{r}\le\overline{r}_2$, we can write
$$\left|\frac{\overline{u}_1'(\overline{r})}{\overline{r}}\right|\le\frac{\lambda^{q_2}M_1^{2-p}}{(n-p)}\overline{r}_2^{m_2+1}. \qquad (46)$$ {eq2.40}

Similarly, we obtain
$$\left|\frac{\overline{U}_2'(\overline{r})}{\overline{r}}\right|\le\frac{M_0^{q_1}M_2^{2-p}}{\lambda^{p-1}(n-p)}\overline{r}_2^{m_1+1}. \qquad (47)$$ {eq2.41}

Further, using estimates (45)–(47), from (35) and (36), we obtain estimates for the second derivatives $\overline{u}_i'(\overline{r})$, $i=1,2$. Therefore,
$$\|\overline{u}_i\|_{C^2[0,\overline{r}_i]}\le K_i,\qquad i=1,2, \qquad (48)$$ {eq2.42}

where K_i depend only on λ, $\overline{r}_2$, m_1, m_2, n, q_1, q_2.

Thus, we have proved the relations $\overline{u}_i(\overline{r}_i)=0$, $\overline{u}_i(\overline{r})>0$ for $\overline{r}\in[0,\overline{r}_i)$ and $\overline{u}_i\in C^2[0,\overline{r}_i]$. The lemma is proved. □

By this lemma, to each value of λ correspond unique values of $\overline{r}_1(\lambda)$, $\overline{r}_2(\lambda)$ such that $\overline{u}_i(\overline{r}_i(\lambda))=0$, $i=1,2$. Thus, we have determined the functions $\overline{r}_1(\lambda)$ and $\overline{r}_2(\lambda)$.

{lem2}

Lemma 2. *The functions $\overline{r}_i(\lambda)$, $(i=1,2)$ are continuous, the function $\overline{r}_1(\lambda)$ decreases, while the function $\overline{r}_2(\lambda)$ increases for $\lambda>0$; further,*
$$\lim_{\lambda\to 0}\overline{r}_1(\lambda)=\infty,\qquad\lim_{\lambda\to\infty}\overline{r}_1(\lambda)=0, \qquad (49)$$ {eq2.43}
$$\lim_{\lambda\to 0}\overline{r}_2(\lambda)=0,\qquad\lim_{\lambda\to\infty}\overline{r}_2(\lambda)=\infty. \qquad (50)$$ {eq2.44}

Proof. Multiplying Eqs. (35), (36) by $\overline{r}^{(n-1)/(p-1)}$ and twice integrating the resulting equality from 0 to $\overline{r}$, taking the initial conditions (22), (23), and (37) into account, we obtain
$$\overline{u}_1(\overline{r})=1-\frac{\lambda^{q_2}}{n-p}\int_0^{\overline{r}}s^{m_2+(n-1)/(p-1)}(s^{(p-n)/(p-1)}-\overline{r}^{(p-n)/(p-1)}))$$
$$\times|\overline{U}_2(s)|^{q_2}|u_1'(s)|^{2-p}\,ds,$$

$$\overline{U}_2(\overline{r}) = 1 - \frac{1}{\lambda^{p-1}(n-p)} \int_0^{\overline{r}} s^{m_1+(n-1)/(p-1)}(s^{(p-n)/(p-1)} - \overline{r}^{(p-n)/(p-1)})) \times |\overline{u}_1(s)|^{q_1} |u_2'(s)|^{2-p}\, ds.$$

Combining this with the equalities $\overline{u}_1(\overline{r}_1) = \overline{U}_2(\overline{r}_2) = 0$, we can write

$$\lambda^{-q_2} = \frac{1}{n-p} \int_0^{\overline{r}_1} s^{m_2+(n-1)/(p-1)}(s^{(p-n)/(p-1)} - \overline{r}_1^{(p-n)/(p-1)}))|\overline{U}_2(s)|^{q_2} |u_1'(s)|^{2-p}\, ds, \tag{51}$$ {eq2.45}

$$\lambda^{p-1} = \frac{1}{n-p} \int_0^{\overline{r}_2} s^{m_1+(n-1)/(p-1)}(s^{(p-n)/(p-1)} - \overline{r}_2^{(p-n)/(p-1)}))|\overline{u}_1(s)|^{q_1} |U_2'(s)|^{2-p}\, ds. \tag{52}$$ {eq2.46}

Differentiating these equalities with respect to λ, we obtain

$$-q_2\lambda^{-q_2-1} = \overline{r}_1^{1-n} \frac{d\overline{r}_1}{d\lambda} \int_0^{\overline{r}_1} s^{m_2+n-1} |\overline{U}_2(s)|^{q_2}\, ds > 0,$$

$$1 = \overline{r}_2^{1-n} \frac{d\overline{r}_2}{d\lambda} \int_0^{\overline{r}_2} s^{m_1+n-1} |\overline{u}_1(s)|^{q_1}\, ds > 0.$$

This implies

$$\frac{d\overline{r}_1}{d\lambda} = -\frac{\overline{r}_1^{n-1}}{q_2\lambda^{q_2+1} \int_0^{\overline{r}_1} s^{m_2+n-1} |\overline{U}_2(s)|^{q_2}\, ds} < 0,$$

$$\frac{d\overline{r}_2}{d\lambda} = \frac{\overline{r}_2^{1-n}}{\int_0^{\overline{r}_2} s^{m_1+n-1} |\overline{u}_1(s)|^{q_1}\, ds} > 0. \tag{53}$$ {eq2.47}

Therefore, $\overline{r}_1$ decreases, while $\overline{r}_2$ increases with respect to λ. In addition, this implies that $d\overline{r}_1/d\lambda$ and $d\overline{r}_1/d\lambda$ are continuous functions with respect to $\lambda > 0$. In that case, so are the functions $\overline{r}_1(\lambda)$ and $\overline{r}_2(\lambda)$.

Since the integrands $|\overline{U}_2(s)|^{q_2}$, $|u_1'(s)|^{2-p}$ in (51) are bounded for $0 \le \overline{r} \le \overline{r}_1$ in view of (45), it follows that $\overline{r}_1 \to 0$ as $\lambda \to \infty$ and $\overline{r}_1 \to \infty$ as $\lambda \to 0$. In the same way, the integrands $|\overline{u}_1(s)|^{q_1}$, $|U_2'(s)|^{2-p}$ in (52) are bounded for $0 \le \overline{r} \le \overline{r}_2$ in view of (45). Therefore, it follows from (52) that $\overline{r}_2 \to 0$ as $\lambda \to 0$ and $\overline{r}_2 \to \infty$ as $\lambda \to \infty$. The lemma is proved. □

It follows from Lemmas 1 and 2 that the function $\overline{r}_1(\lambda)$ is continuous and monotone decreasing from ∞ to 0, while the function $\overline{r}_2(\lambda)$ is also continuous and monotone increasing from 0 to ∞. Therefore, there exists a unique value of λ_0 such that $\overline{r}_1(\lambda_0) = \overline{r}_2(\lambda_0) = \overline{r}_0$, i.e., the equation $\overline{r}_1(\lambda) = \overline{r}_2(\lambda)$ has a unique positive solution $\lambda = \lambda_0$. Therefore, the following statement holds.

{lem3}

Lemma 3. *There exists a unique value of* $\lambda_0 > 0$ *that corresponds to the unique value of* $\overline{r}_0 > 0$ *for which the Cauchy problem* (13), (14), (22), (23) *has a unique solution* $\overline{u} = (\overline{u}_1, \overline{u}_2)$ *such that* $\overline{u}_1(\overline{r}_0) = \overline{u}_2(\overline{r}_0) = 0$ *and* $\overline{u}_i(\overline{r}) > 0$ *for* $\overline{r} \in [0, \overline{r}_0)$, $i = 1, 2$.

The following statement holds.

{th1}

Theorem 1. *If* $1 < p \le 2$ *for* $n \ge 3$ *and* $1 < p < 2$, *for* $n = 2$, *and conditions* (24) *hold, then the Dirichlet problem* (1)–(3) *has a unique* positive radially symmetric solution.

Proof. By Lemma 3, there exists a unique value of $\lambda > 0$ that corresponds to the unique value of $\overline{r}_0 > 0$ for which the Cauchy problem (13), (14), (22), (23) has a unique positive solution $\overline{u} = (\overline{u}_1, \overline{u}_2) \in C^2[0, \overline{r}_0]$ such that $\overline{u}_1(\overline{r}_0) = \overline{u}_2(\overline{r}_0) = 0$ and $\overline{u}_i(\overline{r}) > 0$ for $\overline{r} \in [0, \overline{r}_0)$.

We choose the transformation parameter A in (10) so that the value of $\overline{r} = \overline{r}_0$ corresponds to that of $r = 1$, i.e., from the equality

$$A^{\alpha}\overline{r}_0 = 1; \tag{54}$$ {eq2.48}

then, in view of (9), $u_1(1) = u_2(1) = 0$. From (54), A is uniquely determined:

$$A = (\overline{r}_0)^{-1/\alpha}, \tag{55}$$ {eq2.49}

where α is determined by equality (21). Then, in view of (18),

$$B = A^{\beta_2}\,\overline{u}_2(0) = A^{\beta_2}\lambda_0,$$

with β_2 from (20) is also uniquely determined.

Thus, the solution of the boundary-value problem (7)–(9) is unique, because it coincides with unique solution $u = (u_1, u_2) \in C^2[0,1]$ of the Cauchy problem for system (7), (8) with initial conditions $u_1(0) = A$, $u_2(0) = B$, $u_1'(0) = u_2'(0) = 0$. Therefore, the Dirichlet problem (1)–(3) has a unique positive radially symmetric solution. The theorem is proved. □

{rem1}

Remark 1. Note that if $m_1 = m_2 = m$, $q_1 = q_2 = q$, then, it follows from this theorem that, under the conditions $1 < q \leq (n+m)/(n-2)$, problem (1)–(3) has a unique positive radially symmetric solution that, obviously, coincides with the unique positive radially symmetric solution for one equation $\Delta u + |x|^m |u|^q = 0$ (see [11]).

{rem2}

Remark 2. Note that if $p = 2$ for $n = 2$, then, as proved in [12], for the existence and uniqueness of a positive radially symmetric solution, constraints of type (24) are not required; it suffices to set $q_i > 1$, $i = 1, 2$.

{rem3}

Remark 3. Since the change $x = Rt$ takes the ball of arbitrary radius R: $S_R = \{x \in \mathbb{R}^n : |x| \leq R\}$ to the ball of radius 1: $S_1 = \{t \in \mathbb{R}^n : |t| \leq 1\}$, it follows that Theorem 1 also holds for the ball S_R of arbitrary radius $R > 0$.

3. NONEXISTENCE OF A GLOBAL SOLUTION FOR $n \geq 2$

{ssec3}

Let us show that conditions (24) ensure not only the existence of a unique positive radially symmetric solution of the Dirichlet problem for system (1), (2), but also the nonexistence of a global positive solution of this system. To prove the nonexistence of a global positive solution, Pokhozhaev (see, for example, [1]) and others successfully applied the method of test functions. In [7], Zou conjectured that the existence of such a solution in a bounded domain is equivalent to the nonexistence of a nontrivial global solution. In addition, he noted that an a priori estimate of the solution can ensure both the existence of a positive solution of the Dirichlet problem and the nonexistence of a global solution. To prove the nonexistence of a global positive solution, we shall use the a priori estimates (45) from the previous section.

Let R be an arbitrary positive number, and let $S_R = \{x \in \mathbb{R}^n, |x| \leq R\}$ be a ball with boundary Γ_R. Consider the Dirichlet problem for system (1), (2) with boundary condition

$$u_i|_{\Gamma_R} = 0, \qquad i = 1, 2. \tag{56}$$ {eq3.1}

It is easy to verify that by replacing

$$y = \frac{x}{R}, \qquad u_i = \alpha_i v_i \tag{57}$$ {eq3.2}

and putting $\alpha = R^{-\gamma_i}$, where

$$\gamma_1 = \frac{(p+m_1)q_2 + (p+m_2)(p-1)}{q_1 q_2 - (p-1)^2} > 0, \qquad \gamma_2 = \frac{(p+m_2)q_1 + (p+m_1)(p-1)}{q_1 q_2 - (p-1)^2} > 0,$$

the Dirichlet problem for system (1), (2) with boundary conditions (56) reduces to the Dirichlet problem for the similar system

$$\Delta_p v_1 + |y|^{m_2} |v_2|^{q_2} = 0, \qquad x \in S,$$

$$\Delta_p v_2 + |y|^{m_1}|v_1|^{q_1} = 0, \qquad x \in S,$$

in the unit ball S with boundary condition

$$u_i|_{\Gamma=0}, \qquad i = 1, 2.$$

In view of (10) and (45), the positive radially symmetric solution of problem (1)–(3) satisfies the estimates

$$|u_1| = A^{\beta_1}|\overline{u}_1| \le A^{\beta_1}M_0, \qquad |u_2| = A^{\beta_2}|\overline{u}_2| = A^{\beta_2}|\lambda_0\overline{U}_2| \le A^{\beta_2}\lambda_0,$$

where A, β_1, β_2 are defined by the equalities (55), (19), (20), respectively; the existence of λ_0 and r_0 was proved in Lemma 2. Then it follows from (57) that thepositive radially symmetric solution of the Dirichlet problem for system (1), (2) in the ball of arbitrary radius R with boundary condition (56) satisfies the estimates

$$0 \le u_1 \le \frac{1}{R^{\gamma_1}}, \qquad 0 \le u_2 \le \frac{\lambda_0}{R^{\gamma_2}}.$$

Hence, letting $R \to \infty$, we see that the following statement is valid.

{th2}

Theorem 2. *If $1 < p \le 2$ for $n \ge 3$ and $1 < p < 2$ for $n = 2$ and conditions* (24) *hold, then, for $n \ge 0$, the Dirichlet of problem* (1)–(3) *has no global* positive radially symmetric solution.

4. EXISTENCE AND UNIQUENESS OF A POSITIVE RADIALLY SYMMETRIC SOLUTION FOR $n = 1$

{ssec4}

In this case, the radially symmetric solution of problem (1)–(3) satisfies the system

$$(p-1)u_1'' = -r^{m_2}|u_2|^{q_2}|u_1'|^{2-p}, \qquad 0 < r < 1, \tag{58}$$ {eq4.1}

$$(p-1)u_2'' = -r^{m_1}|u_1|^{q_1}|u_2'|^{2-p}, \qquad 0 < r < 1, \tag{59}$$ {eq4.2}

and the boundary conditions

$$u_i'(0) = 0, \quad u_i(1) = 0, \qquad i = 1, 2. \tag{60}$$ {eq4.3}

Just as in Sec. 1, by the transformation (10), system (58), (59) reduces to the invariant form

$$(p-1)\overline{u}_1'' = -\overline{r}^{m_2}|\overline{u}_2|^{q_2}|\overline{u}_1'|^{2-p}, \tag{61}$$ {eq4.4}

$$(p-1)\overline{u}_2'' = -\overline{r}^{m_1}|\overline{u}_1|^{q_1}|\overline{u}_2'|^{2-p} \tag{62}$$ {eq4.5}

with initial conditions

$$\overline{u}_1(0) = 1, \qquad \overline{u}_2(0) = \lambda > 0, \tag{63}$$ {eq4.6}

$$\overline{u}_1'(0) = \overline{u}_2'(0) = 0. \tag{64}$$ {eq4.7}

Since, in view of (61), (62), $\overline{u}_i''$ are negative for $\overline{r} > 0$, it follows that the functions $\overline{u}_i$, $i = 1, 2$, are convex (upward). Integrating Eqs. (61), (62) and taking into account the initial conditions (64), we obtain

$$\overline{u}_1'(\overline{r}) = -\frac{1}{p-1}\int_0^{\overline{r}} s^{m_2}|\overline{u}_2(s)|^{q_2}|\overline{u}_1'(s)|^{2-p}\,ds, \tag{65}$$ {eq4.8}

$$\overline{u}_2'(\overline{r}) = -\frac{1}{p-1}\int_0^{\overline{r}} s^{m_1}|\overline{u}_1(s)|^{q_1}|\overline{u}_2'(s)|^{2-p}\,ds. \tag{66}$$ {eq4.9}

This implies that the functions $\overline{u}_i(\overline{r})$ decrease for $\overline{r} > 0$. Thus, the functions $\overline{u}_i(\overline{r})$, $i = 1, 2$, are decreasing and convex (upward). Since $\overline{u}_i(0) > 0$, there exist unique points $\overline{r}_i > 0$ such that $\overline{u}_i(\overline{r}_i) = 0$, $i = 1, 2$.

So we have proved the following statement.

{lem4}

Lemma 4. *For an arbitrary positive number λ, for $1 < p \le 2$, there exist unique positive numbers $\overline{r}_1$, $\overline{r}_2$ for which the Cauchy problem (61)–(64) has a unique positive solution $\overline{u} = (\overline{u}_1, \overline{u}_2)$, $\overline{u}_i \in C^2[0, \overline{r}_i]$, $\overline{u}_i(\overline{r}_i) = 0$, $\overline{u}_i(\overline{r}) > 0$ for $\overline{r} \in [0, \overline{r}_i)$, $i = 1, 2$.*

By this lemma, to each value of $\lambda > 0$ correspond unique values of $\overline{r}_1(\lambda)$, $\overline{r}_2(\lambda)$ such that $\overline{u}_i(\overline{r}_i(\lambda)) = 0$, $i = 1, 2$, i.e., the functions $\overline{r}_i(\lambda)$, $i = 1, 2$, are defined.

{lem5}

Lemma 5. *The functions $\overline{r}_i(\lambda)$, $i = 1, 2$, are continuous, $\overline{r}_1(\lambda)$ decreases, while $\overline{r}_2(\lambda)$ increases for $\lambda > 0$; further,*

$$\lim_{\lambda \to 0} \overline{r}_1(\lambda) = \infty, \qquad \lim_{\lambda \to \infty} \overline{r}_1(\lambda) = 0, \tag{67}$$ {eq4.10}

$$\lim_{\lambda \to 0} \overline{r}_1(\lambda) = 0, \qquad \lim_{\lambda \to \infty} \overline{r}_1(\lambda) = \infty. \tag{68}$$ {eq4.11}

Proof. In Eqs. (61), (62), let us make the change $\overline{u}_2 = \lambda \overline{U}_2$. Then these equations take the form

$$(p-1)\overline{u}_1'' = -\lambda^{q_2} \overline{r}^{m_2} |\overline{U}_2|^{q_2} |\overline{u}_1'|^{2-p}, \tag{69}$$ {eq4.12}

$$(p-1)\overline{u}_2'' = -\overline{r}^{m_1} |\overline{u}_1|^{q_1} |\overline{U}_2'|^{2-p}. \tag{70}$$ {eq4.13}

It follows from the conditions $\overline{u}_2(0) = \lambda$, $\overline{u}_2'(0) = 0$ that $\overline{U}_2(0) = 1$, $\overline{U}_2'(0) = 0$. Integrating Eqs. (69), (70) from 0 to $\overline{r}$ and taking the initial conditions into account, we obtain

$$\overline{u}_1'(\overline{r}) = -\frac{\lambda^{q_2}}{p-1} \int_0^{\overline{r}} s^{m_2} |\overline{U}_2(s)|^{q_2} |\overline{u}_1'(s)|^{2-p}\, ds, \tag{71}$$ {eq4.14}

$$\overline{U}_2'(\overline{r}) = -\frac{1}{\lambda^{p-1}(p-1)} \int_0^{\overline{r}} s^{m_1} |\overline{u}_1(s)|^{q_1} |\overline{U}_2'(s)|^{2-p}\, ds. \tag{72}$$ {eq4.15}

This implies that $\overline{u}_1$ and $\overline{U}_2$ decrease. Integrating the resulting equalities and taking the initial conditions into account, we can write

$$\overline{u}_1(\overline{r}) = -\frac{\lambda^{q_2}}{p-1} \int_0^{\overline{r}} s^{m_2} (\overline{r} - s) |\overline{U}_2(s)|^{q_2} |\overline{u}_1'(s)|^{2-p}\, ds, \tag{73}$$ {eq4.16}

$$\overline{U}_2(\overline{r}) = -\frac{1}{\lambda^{p-1}(p-1)} \int_0^{\overline{r}} s^{m_1} (\overline{r} - s) |\overline{u}_1(s)|^{q_1} |\overline{U}_2'(s)|^{2-p}\, ds. \tag{74}$$ {eq4.17}

This implies the inequalities $0 \le \overline{u}_1(\overline{r}) \le 1$, $0 \le \overline{u}_2(\overline{r}) \le \lambda$.

Since $\overline{u}_1(\overline{r}_1) = 0$, $\overline{U}_2(\overline{r}_2) = 0$, using (73), (74) we obtain

$$\lambda^{-q_2} = \frac{1}{p-1} \int_0^{\overline{r}_1} s^{m_2} (\overline{r}_1 - s) |\overline{U}_2(s)|^{q_2} |\overline{u}_1'(s)|^{2-p}\, ds, \tag{75}$$ {eq4.18}

$$\lambda^{p-1} = \frac{1}{p-1} \int_0^{\overline{r}_2} s^{m_1} (\overline{r}_2 - s) |\overline{u}_1(s)|^{q_1} |\overline{U}_2'(s)|^{2-p}\, ds. \tag{76}$$ {eq4.19}

Denote

$$\max_{0 \le \overline{r} \le \overline{r}_1} |\overline{u}_1(\overline{r})| = a, \qquad \max_{0 \le \overline{r} \le \overline{r}_2} |\overline{U}_2(\overline{r})| = b.$$

Using (71), (72) and taking into account the inequalities $0 \le \overline{u}_1(\overline{r}) \le 1$ for $\overline{r} \in [0, \overline{r}_1]$ and $0 \le \overline{U}_2(\overline{r}) \le 1$ for $\overline{r} \in [0, \overline{r}_2]$, we obtain

$$a \le \frac{\lambda^{q_2}}{p-1} a^{2-p} \frac{\overline{r}_1^{m_2+1}}{m_2+1}, \qquad b \le \frac{1}{\lambda^{p-1}(p-1)} b^{2-p} \frac{\overline{r}_2^{m_1+1}}{m_1+1}.$$

This implies

$$a \le \lambda^{q_2/(p-1)} \overline{r}_1^{(m_2+1)/(p-1)} A, \tag{77}$$ {eq4.20}

$$b \le \frac{1}{\lambda} \overline{r}_2^{(m_1+1)/(p-1)} B, \tag{78}$$ {eq4.21}

where $A = [(p-1)(m_2+1)]^{-1/(p-1)}$, $B = [(p-1)(m_1+1)]^{-1/(p-1)}$. Then, using (75), we can write

$$\frac{1}{\lambda^{q_2/(p-1)}} \le \frac{A^{2-p} \overline{r}_1^{(m_2+1)/(p-1)}}{(p-1)(m_2+1)(m_2+2)}.$$

This implies

$$\overline{r}_1^{(m_2+p)/(p-1)} \ge \frac{(p-1)(m_2+1)(m_2+2)}{A^{2-p} \lambda^{q_2/(p-1)}}. \tag{79}$$ {eq4.22}

It follows from (76) that

$$\lambda \le \frac{B^{2-p} \overline{r}_2^{(m_1+p)/(p-1)}}{(p-1)(m_1+1)(m_1+2)}.$$

Hence we have

$$r_2^{(m_1+p)/(p-1)} \ge \frac{\lambda(p-1)(m_1+1)(m_1+2)}{B^{2-p}}. \tag{80}$$ {eq4.23}

It follows from (79) and (80) that

$$\lim_{\lambda \to \ 0} \overline{r}_1(\lambda) = \infty, \qquad \lim_{\lambda \to \infty} \overline{r}_2(\lambda) = \infty.$$

Since the integrand on the right-hand side of (24) is greater than zero for $\overline{r}_1 > 0$, passing to the limit as $\lambda \to \infty$ in this equality, we obtain $\lim_{\lambda \to \infty} \overline{r}_1(\lambda) = 0$. Similarly, passing to the limit as $\lambda \to \ 0$ in equality (25), we obtain $\lim_{\lambda \to \ 0} \overline{r}_1(\lambda) = 0$.

Equalities (67), (68) hold. The lemma is proved. $\square$

It follows from Lemmas 4 and 5 that the function $\overline{r}_1(\lambda)$ is continuous and decreases from ∞ to 0, while the function $\overline{r}_2(\lambda)$ is also continuous and increases from 0 to ∞. Therefore, the unique value of λ_0 satisfies the equalities $\overline{r}(\lambda_0) = \overline{r}_2(\lambda_0) = \overline{r}_0$, i.e., the equation $\overline{r}(\lambda) = \overline{r}_2(\lambda)$ has a unique positive solution $\lambda = \lambda_0$. Therefore, the following statement is valid.

{lem6}

Lemma 6. *There exists a unique value of* $\lambda_0 > 0$ *that corresponds to the unique value of* $\overline{r}_0 > 0$ *such that the Cauchy problem* (61)–(64) *has a unique solution* $\overline{u} = (\overline{u}_1, \overline{u}_2) \in C^2[0, \overline{r}_0]$ *such that* $\overline{u}_1(\overline{r}_0) = \overline{u}_2(\overline{r}_0) = 0$ *and* $\overline{u}_i(\overline{r}) > 0$, $i = 1, 2$, *for* $\overline{r} \in [0, \overline{r}_0)$.

The following statement is valid.

{th3}

Theorem 3. *For* $n = 1$, $q_i > 1$, $i = 1, 2$, *the Dirichlet problem* (1)–(3) *has a unique* positive radially symmetric solution.

{th4}

Theorem 4. *For* $n = 1$, $q_i > 1$, $i = 1, 2$, *the Dirichlet problem* (1)–(3) *has no global* positive radially symmetric solution.

The proof of both these theorems is similar to that of Theorems 1 and 2, respectively.

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